Aufgabe 1

(a) $\mathrm{d}E = mv\,\mathrm{d}v$

(b) $\mathrm{d}y = y_0\omega\cos(\omega t + \varphi_0)\,\mathrm{d}t$

(c) $$\begin{aligned} f(v) &= f_0\frac{1}{2}\left(\frac{c-v}{c+v}\right)^{-\frac{1}{2}} \cdot \frac{(-1)\cdot(c+v)-(c-v)\cdot 1}{(c+v)^2}\,\mathrm{d}v \\ &= f_0\frac{1}{2}\cdot\frac{\sqrt{c+v}}{\sqrt{c-v}}\cdot\frac{-2c}{(c+v)^2}\,\mathrm{d}v \\ &= -cf_0\frac{\sqrt{c+v}}{\sqrt{c-v}}\cdot\frac{1}{(c+v)^2}\,\mathrm{d}v \\ &= \frac{-cf_0}{\sqrt{c-v}\sqrt{(c+v)^3}}\,\mathrm{d}v = \frac{-cf_0}{\sqrt{(c-v)(c+v)^3}}\,\mathrm{d}v \\ &= \frac{-cf_0}{\sqrt{(c^2-v^2)(c+v)}}\,\mathrm{d}v \end{aligned}$$

Aufgabe 2

(a) $x = t - 1 \quad\Rightarrow\quad t = x + 1$
$y = 1 - t^2$

$$y = 1 - t^2 = 1 - (x+1)^2 = 1 - (x^2 + 2x + 1) = -x^2 - 2x$$

(b) $\dot{x}(t) = 1$
$\dot{y}(t) = -2t$

$$\begin{aligned} s &= \int_0^2 \sqrt{1^2 + (-2t)^2}\,\mathrm{d}t = \int_0^2 \sqrt{1+4t^2}\,\mathrm{d}t \\ &= 2\int_0^2 \sqrt{\frac{1}{4}+t^2}\,\mathrm{d}t = 2\left[\frac{t}{2}\sqrt{\frac{1}{4}+t^2} + \frac{1}{8}\ln\left(t + \sqrt{\frac{1}{4}+t^2}\right)\right]_0^2 \\ &= 2\left(\frac{\sqrt{17}}{2} + \frac{1}{8}\ln\left(2 + \frac{\sqrt{17}}{2}\right)\right) - 2\left(\frac{1}{8}\ln\left(\frac{1}{2}\right)\right) \\ &= \sqrt{17} + \frac{1}{4}\ln\left(2+\frac{\sqrt{17}}{2}\right) - \frac{1}{4}\ln\left(\frac{1}{2}\right) \\ &= \sqrt{17} + \frac{1}{4}\ln\left(4+\sqrt{17}\right) \approx 4.647 \end{aligned}$$

Aufgabe 3

$$f(x) = \frac{1}{2}\left(\frac{x^3}{3} + \frac{1}{x}\right) \quad \Rightarrow \quad f'(x) = \frac{1}{2}\left(x^2 - \frac{1}{x^2}\right)$$

$$\sqrt{1 + \left[f'(x)\right]^2} = \sqrt{\frac{4}{4} + \frac{1}{4}\left(x^4 - 2 + \frac{1}{x^4}\right)} = \sqrt{\frac{1}{4}\left(x^4 + 2 + \frac{1}{x^4}\right)}$$

$$= \sqrt{\frac{1}{4}\left(x^2 + \frac{1}{x^2}\right)^2} = \frac{1}{2}\left(x^2 + \frac{1}{x^2}\right)$$

$$\int_1^2 \sqrt{1 + \left[f'(x)\right]^2}\,\mathrm{d}x = \frac{1}{2}\int_1^3 \left(x^2 + \frac{1}{x^2}\right) = \frac{1}{2}\left[\frac{1}{3}x^3 - \frac{1}{x}\right]_1^2$$

$$= \frac{1}{2}\left(\frac{8}{3} - \frac{1}{2} - \frac{1}{3} + 1\right) = \frac{17}{12} \approx 1.417$$

Aufgabe 4

$$r(\varphi) = \frac{2}{\varphi} \quad \Rightarrow \quad r'(\varphi) = -\frac{2}{\varphi^2}$$

$$s = \int_1^{2\pi} \sqrt{\left[r(\varphi)\right]^2 + \left[r'(\varphi)\right]^2}\,\mathrm{d}\varphi = \int_1^{2\pi} \sqrt{\frac{4}{\varphi^2} + \frac{4}{\varphi^4}}\,\mathrm{d}\varphi$$

$$= \int_1^{2\pi} \sqrt{\frac{4\varphi^2 + 4}{\varphi^4}}\,\mathrm{d}\varphi = 2\int_1^{2\pi} \frac{\sqrt{\varphi^2 + 1}}{\varphi^2}\,\mathrm{d}\varphi$$

$$= 2\left[\operatorname{arcsinh}(\varphi) - \frac{\sqrt{\varphi^2 + 1}}{\varphi}\right]_1^{2\pi} \approx 4.115$$

Wen es interessiert, kann die folgende kleingedruckte Herleitung studieren.

$$\int \frac{\sqrt{x^2+1}}{x^2}\,\mathrm{d}x = \ldots \qquad f'(x) = \frac{1}{x^2} \quad \Rightarrow \quad f(x) = -\frac{1}{x}$$

$$g(x) = \sqrt{x^2+1} \quad \Rightarrow \quad g'(x) = \frac{2x}{2\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}}$$

$$\cdots = -\frac{\sqrt{x^2+1}}{x} - \int \left(-\frac{1}{x}\right)\frac{x}{\sqrt{x^2+1}}\,\mathrm{d}x = \int \frac{1}{\sqrt{x^2+1}}\,\mathrm{d}x - \frac{\sqrt{x^2+1}}{x}$$

$$= \ln\left(x + \sqrt{x^2+1}\right) - \frac{\sqrt{x^2+1}}{x} + C = \operatorname{arcsinh}(x) - \frac{\sqrt{x^2+1}}{x} + C$$

Die Herleitung der zweiten Stammfunktion ist einfacher und kürzer:

Beachte: $\sinh(x) = \frac{1}{2}\left(\mathrm{e}^x - \mathrm{e}^{-x}\right) \quad \Rightarrow \quad [\sinh(x)]' = \ldots = \cosh(x)$

$\cosh(x) = \frac{1}{2}\left(\mathrm{e}^x + \mathrm{e}^{-x}\right) \quad \Rightarrow \quad [\cosh(x)]' = \ldots = \sinh(x)$

$\sinh^2(x) + \cos^2(x) = \ldots = -1$

$$\int \frac{1}{\sqrt{x^2+1}}\,\mathrm{d}x = \ldots \qquad x \overset{\text{Sub}}{=} \sinh(t) \quad \Rightarrow \quad \mathrm{d}x = \cosh(t)\,\mathrm{d}t$$

$$= \int \frac{\cosh(t)}{\sqrt{\sinh^2(t) + 1}}\,\mathrm{d}t = \int \frac{\cosh(t)}{\sqrt{\cosh^2(t)}}\,\mathrm{d}t = \int 1\,\mathrm{d}t = t \overset{\text{Sub}^{-1}}{=} \operatorname{arcsinh}(x)$$

Aufgabe 5

$$r(\varphi) = 1 + \sin(4\varphi) \quad \Rightarrow \quad r'(\varphi) = 4\cos(4\varphi)$$

$$s = \int_0^{2\pi} \sqrt{\big[r(\varphi)\big]^2 + \big[r'(\varphi)\big]^2}\,\mathrm{d}\varphi$$

$$= \int_0^{2\pi} \sqrt{\big(1 + \sin 4\varphi\big)^2 + \big(4\cos 4\varphi\big)^2}\,\mathrm{d}\varphi \approx 18.13$$

Aufgabe 6

$$\int_3^8 \frac{x}{\sqrt{1+x}}\,\mathrm{d}x = \dots \qquad x(t) = t^2 - 1 \quad \Leftrightarrow \quad t = \sqrt{x+1}$$

$$\mathrm{d}x = 2t\,\mathrm{d}t$$

$$= \int_{\sqrt{3+1}}^{\sqrt{8+1}} \frac{x}{\sqrt{1+x}}\,\mathrm{d}x = \int_2^3 \frac{t^2-1}{\sqrt{1+t^2-1}} \cdot 2t\,\mathrm{d}t$$

$$= \int_2^3 \frac{t^2-1}{t} \cdot 2t\,\mathrm{d}t = 2\int_2^3 (t^2-1)\,\mathrm{d}t = 2\left[\frac{1}{3}t^3 - t\right]_2^3$$

$$= 2\left(9 - 3 - \frac{8}{3} + 2\right) = \frac{32}{3}$$

Aufgabe 7

$$\int \frac{1}{x^2+2x+2}\,\mathrm{d}x = \dots \qquad x = t - 1 \quad \Leftrightarrow \quad t \overset{(*)}{=} x + 1$$

$$\mathrm{d}x = \mathrm{d}t$$

$$= \int \frac{1}{x^2+2x+2}\,\mathrm{d}x \overset{(1)}{=} \int \frac{1}{(t-1)^2 + 2(t-1) + 2}\,\mathrm{d}t$$

$$= \int \frac{1}{t^2 - 2t + 1 + 2t - 2 + 2}\,\mathrm{d}t = \int \frac{1}{t^2+1}\,\mathrm{d}t$$

$$\overset{\text{FS}}{=} \arctan t \overset{(*)}{=} \arctan(x+1) + C$$

Aufgabe 8

$I = 2\,\mathrm{mA} \cdot (1 \pm 2\%)$

$R = 12\,\mathrm{k\Omega} \cdot (1 \pm 5\%)$

$\Delta I = 0.02 \cdot 2\,\mathrm{mA} = 0.04\,\mathrm{mA}$

$\Delta R = 0.6 \cdot 12\,\mathrm{k\Omega} = 0.6\,\mathrm{k\Omega}$

$U = I \cdot R = 2\,\mathrm{mA} \cdot 12\,\mathrm{k\Omega} = 24\,\mathrm{V}$

$$\Delta U = \left|\frac{\partial U}{\partial I} \cdot \Delta I\right| + \left|\frac{\partial U}{\partial R} \cdot \Delta R\right| = |R \cdot \Delta I| + |I \cdot \Delta R|$$

$$= 12\,\mathrm{k\Omega} \cdot 0.04\,\mathrm{mA} + 2\,\mathrm{mA} \cdot 0.6\,\mathrm{k\Omega} = 1.68\,\mathrm{V}$$

$$r_U = \frac{\Delta U}{U} = \frac{1.68\,V}{24\,V} = 0.07$$

$U = 24\,\mathrm{V} \cdot (1 \pm 7\%)$

Aufgabe 9

$c = (37 \pm 2)\,\mathrm{cm}$

$$\alpha = (42 \pm 1)^\circ = \left(\frac{42\pi}{180} \pm \frac{1\pi}{180}\right)$$

$a = c \cdot \sin(\alpha) = 25\,\mathrm{cm}$

$$\Delta a = \left|\frac{\partial a}{\partial c} \cdot \Delta c\right| + \left|\frac{\partial a}{\partial \alpha} \cdot \Delta \alpha\right| = |\sin(\alpha) \cdot \Delta c| + |c\cos(\alpha) \cdot \Delta\alpha|$$

$$= \sin\left(\frac{42\pi}{180}\right) \cdot 2\,\mathrm{cm} + \cos\left(\frac{42\pi}{180}\right) \cdot 37\mathrm{cm} = 1.8\,\mathrm{cm}$$

$a = (25 \pm 2)\,\mathrm{cm} = 25\,\mathrm{cm} \pm 8\%$