
Integralrechnung
Übungen (L+)

Aufgabe 1.1

(a) $\int_2^5 (2x + 3) dx = 30$

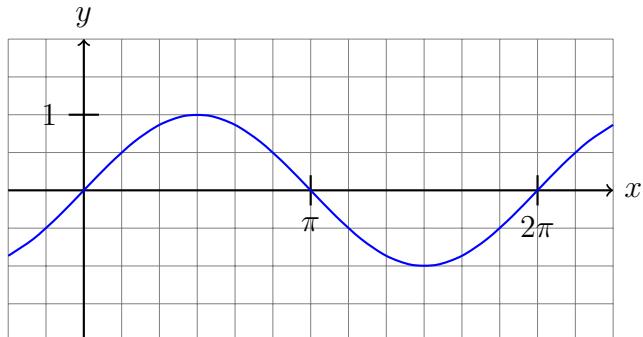
(b) $\int_0^4 \sqrt{x} dx \approx 5.333$

(c) $\int_0^1 e^x dx \approx 1.718$

(d) $\int_1^e \ln(x) dx \approx 0.434$

Aufgabe 1.2

(a) Graph von $y = \sin x$:



Aufgabe 1.2

(b) • $\int_0^\pi \sin x dx = 2$

• $\int_\pi^{2\pi} \sin x dx = -2$

• $\int_0^{2\pi} \sin x dx = 0$

(c) Da die Flächen von 0 bis π bzw. von π bis 2π verfahrensbedingt unterschiedliche „Vorzeichen“ aber gleichen Betrag haben, heben sie sich gegenseitig auf.

(d) $A = 2 + |-2| = 4$ FE (Flächeneinheiten)

Aufgabe 1.3

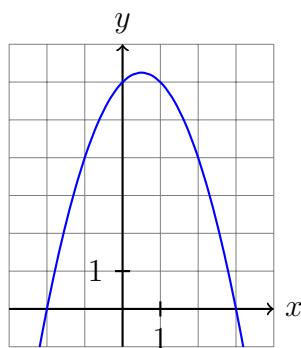
Vertauscht man die Grenzen im bestimmten Integral so ändert sich das Vorzeichen des Resultats:

$$(a) \int_1^2 \frac{1}{x} dx \approx 0.693$$

$$(b) \int_2^1 \frac{1}{x} dx \approx -0.693$$

Aufgabe 1.4

Graph: ($y = -x^2 + x + 6$)



Nullstellen: $x_1 = -2$ (untere Grenze), $x_2 = 3$ (obere Grenze)

$$\text{Flächeninhalt: } A = \int_{-2}^3 (-x^2 + x + 6) dx \approx 20.83$$

Aufgabe 2.1

$$\int x^{10} dx = \frac{1}{11} x^{11} + C$$

Aufgabe 2.2

$$\int e^x dx = e^x + C$$

Aufgabe 2.3

$$\int \frac{1}{x^2} dx = \int x^{-2} dx = -x^{-1} + C = -\frac{1}{x} + C$$

Aufgabe 2.4

$$\int \tan x dx = -\ln |\cos x| + C$$

Aufgabe 2.5

$$\int \sqrt{x} dx = \frac{2}{3} x^{3/2} + C$$

Aufgabe 2.6

$$\int \sin x dx = -\cos x + C$$

Aufgabe 2.7

$$\int 1 dx = x + C$$

Aufgabe 2.8

$$\int \frac{1}{x} dx = \ln |x| + C$$

Aufgabe 2.9

$$\int \ln |x| dx = x(\ln |x| - 1) + C$$

Aufgabe 2.10

$$\int \cos x dx = \sin x + C$$

Aufgabe 2.11

$$\int (x^2 + x) \, dx = \frac{1}{3}x^3 + \frac{1}{2}x^2 + C$$

Aufgabe 2.12

$$\int (\sin x + \cos x) \, dx = -\cos x + \sin x + C$$

Aufgabe 2.13

$$\int (x^5 - x^3 + 2) \, dx = \frac{1}{6}x^6 - \frac{1}{4}x^4 + 2x + C$$

Aufgabe 2.14

$$\int (x-3)(x+4) \, dx = \int (x^2 + x - 12) \, dx = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 12x + C$$

Achtung: Integrale dürfen nicht mit Produkten/Quotienten/Potenzen vertauscht werden! Daher Produkte möglichst als Summen darstellen.

Aufgabe 2.15

$$\begin{aligned} \int \frac{x^2 + x}{x^2} \, dx &= \int \left(\frac{x^2}{x^2} + \frac{x}{x^2} \right) \, dx = \int 1 \, dx + \int \frac{1}{x} \, dx \\ &= x + \ln|x| + C \end{aligned}$$

Aufgabe 2.16

$$\int (\mathrm{e}^x + \mathrm{e}^{-x}) \, dx = \mathrm{e}^x - \mathrm{e}^{-x} + C$$

Aufgabe 2.17

$$\int 5x^4 \, dx = 5 \cdot \frac{1}{5}x^5 = x^5 + C$$

Aufgabe 2.18

$$\int -4 \sin x \, dx = -4 \int \sin x \, dx = -4(-\cos x) = 4 \cos x + C$$

Aufgabe 2.19

$$\int 7 \, dx = 7 \int 1 \, dx = 7x + C$$

Aufgabe 2.20

$$\int (x-2)^2 \, dx = \int (x^2 - 4x + 4) \, dx = \frac{1}{3} x^3 - 2x^2 + 4x + C$$

Aufgabe 2.21

$$\int \frac{3}{x} \, dx = 3 \int \frac{1}{x} \, dx = 3 \ln|x| + C$$

Aufgabe 2.22

$$\int \frac{1}{3} e^{3x} \, dx = \frac{1}{3} \int e^{3x} \, dx = \frac{1}{3} \cdot \frac{1}{3} \cdot e^{3x} \, dx = \frac{1}{9} e^{3x} + C$$

Aufgabe 2.23

$$\int \frac{1}{1+x^2} \, dx = \arctan x + C$$

Aufgabe 2.24

$$\int (3 \cos x - \tan x) \, dx = 3 \sin x + \ln|\cos x| + C$$

Aufgabe 2.25

$$\int (2x^4 - 3)^2 \, dx = \int (4x^8 - 12x^4 + 9) \, dx = \frac{4}{9} x^9 - \frac{12}{5} x^5 + 9x + C$$

Aufgabe 2.26

$$\begin{aligned} \int \sqrt{5x} \, dx &= \int \sqrt{5} \cdot \sqrt{x} \, dx \\ &= \sqrt{5} \cdot \frac{2}{3} x^{3/2} \\ &= \frac{2\sqrt{5}}{3} x^{3/2} + C \end{aligned}$$

Aufgabe 2.27

$$\int 2^x \, dx = \frac{2^x}{\ln 2} + C$$

Aufgabe 2.28

$$\begin{aligned}\int \left(\frac{2}{x} + \frac{3}{x^3} \right) dx &= 2 \int \frac{1}{x} dx + 3 \int x^{-3} dx \\ &= 2 \ln|x| - \frac{3}{2}x^{-2} + C\end{aligned}$$

Aufgabe 2.29

$$\int_0^2 3x^2 dx = [x^3]_0^2 = 2^3 - 0^3 = 8$$

Aufgabe 2.30

$$\int_1^3 x^3 dx = \left[\frac{1}{4}x^4 \right]_1^3 = \frac{81}{4} - \frac{1}{4} = \frac{80}{4} = 20$$

Aufgabe 2.31

$$\int_2^4 (x^2 + 3) dx = \left[\frac{1}{3}x^3 + 3x \right]_2^4 = \frac{74}{3}$$

Aufgabe 2.32

$$\int_1^3 (x^2 - 4x)^2 dx = \int_1^3 (x^4 - 8x^3 + 16x^2) dx = \frac{406}{15}$$

Aufgabe 2.33

$$\int_2^4 x^5(2x+5) dx = \int_2^4 (2x^6 + 5x^5) dx = \left[\frac{2}{7}x^7 + \frac{5}{6}x^6 \right]_2^4 = \frac{56032}{7}$$

Aufgabe 2.34

$$\begin{aligned}\int_{-1}^0 (3x^2 - kx + k) dx &= -2 \\ \left[x^3 - \frac{1}{2}kx^2 + kx \right]_{-1}^0 &= -2 \\ (0 - 0 + 0) - \left(-1 - \frac{1}{2}k - k \right) &= -2 \\ 1 + \frac{3}{2}k &= -2 \\ \frac{3}{2}k &= -3 \\ k &= -2\end{aligned}$$

Aufgabe 2.35

$$\int_1^b (x^2 + x + 1) dx = 144$$

$$\left[\frac{1}{3}x^3 + \frac{1}{2}x^2 + x \right]_1^b = 144$$

$$\left(\frac{1}{3}b^3 + \frac{1}{2}b^2 + b \right) - \left(\frac{1}{3} + \frac{1}{2} + 1 \right) = 144$$

$$\frac{1}{3}b^3 + \frac{1}{2}b^2 + b - \frac{875}{6} = 0$$

$$b = 7$$

Aufgabe 2.36

$$\int_0^4 \sqrt{x} dx = \left[\frac{2}{3}x^{3/2} \right]_0^4 = \frac{16}{3}$$

Aufgabe 2.37

$$\int_0^1 5x\sqrt{x} dx = 5 \int_0^1 x^1 \cdot x^{1/2} dx = 5 \int_0^1 x^{3/2} dx = 5 \left[\frac{2}{5}x^{5/2} \right]_0^1 = 2$$

Aufgabe 2.38

$$\int_0^{\pi/2} \cos x dx = [\sin(x)]_0^{\pi/2} = \sin\left(\frac{\pi}{2}\right) - \sin(0) = 1$$

Aufgabe 2.39

$$\begin{aligned} \int_0^{\pi/4} \sin(2x) dx &= \left[-\frac{1}{2} \cos(2x) \right]_0^{\pi/4} \\ &= -\frac{1}{2} \cos\left(2 \cdot \frac{\pi}{4}\right) - \left(-\frac{1}{2} \cos(2 \cdot 0)\right) \\ &= -\frac{1}{2} \cos\left(\frac{\pi}{2}\right) + \frac{1}{2} \cos(0) = 0 + \frac{1}{2} = \frac{1}{2} \end{aligned}$$

Aufgabe 2.40

$$\int_1^2 \frac{2}{x} dx = 2 \int_1^2 \frac{1}{x} dx = 2 [\ln(x)]_1^2 = 2(\ln(2) - \ln(1)) = 2 \ln(2)$$

Aufgabe 2.41

$$\begin{aligned}
\int_2^3 \frac{x^2 + 4x + 3}{x} dx &= \int_2^3 \left(x + 4 + \frac{3}{x} \right) dx \\
&= \left[\frac{1}{2}x^2 + 4x + 3 \ln|x| \right]_2^3 \\
&= \left(\frac{9}{2} + 12 + 3 \ln|3| \right) - \left(4 + 8 + 3 \ln|2| \right) \\
&= \frac{13}{2} + 3(\ln(3) - \ln(2)) \\
&= \frac{13}{2} + 3 \ln\left(\frac{3}{2}\right)
\end{aligned}$$

Aufgabe 2.42

$$\begin{aligned}
f'(x) &= 3x^2 - 4 \\
f(x) &= x^3 - 4x + C \\
f(5) &= 5^3 - 4 \cdot 5 + C = 54 \quad \Rightarrow \quad C = -51 \\
f(x) &= x^3 - 4x - 51
\end{aligned}$$

Aufgabe 2.43

$$\begin{aligned}
g'(x) &= 5 - x \\
g(x) &= 5x - \frac{1}{2}x^2 + C \\
g(-2) &= -10 - 2 + C = -12 + C \\
-g(2) &= -(10 - 2 + C) = -8 - C \\
g(-2) = -g(2) &\quad \Rightarrow \quad -12 + C = -8 - C \quad \Rightarrow \quad C = 2 \\
g(x) &= -\frac{1}{2}x^2 + 5x + 2
\end{aligned}$$

Aufgabe 2.44

$$\begin{aligned}
f''(x) &= 2x \\
f'(x) &= x^2 + C_1 \\
f'(2) = 5 &\quad \Rightarrow \quad 4 + C_1 = 5 \quad \Rightarrow \quad C_1 = 1 \\
f'(x) &= x^2 + 1 \\
f(x) &= \frac{1}{3}x^3 + x + C_2 \\
f(1) = 3 &\quad \Rightarrow \quad \frac{1}{3} + 1 + C_2 = 3 \quad \Rightarrow \quad C_2 = \frac{5}{3} \\
f(x) &= \frac{1}{3}x^3 + x + \frac{5}{3}
\end{aligned}$$

Aufgabe 2.45

$$f''(x) = x^2 - 1$$

$$f'(x) = \frac{1}{3}x^3 - x + C_1$$

$$f'(0) = 4 \quad \Rightarrow \quad C_1 = 4$$

$$f'(x) = \frac{1}{3}x^3 - x + 4$$

$$f(x) = \frac{1}{12}x^4 - \frac{1}{2}x^2 + 4x + C_2$$

$$f(0) = 0 \quad \Rightarrow \quad C_2 = 0$$

$$f(x) = \frac{1}{12}x^4 - \frac{1}{2}x^2 + 4x$$

Aufgabe 3.1

$$\int_0^1 \sqrt{3x+1} dx = \dots$$

$$\text{Substitution: } u = 3x + 1 \Rightarrow \frac{du}{dx} = 3 \Rightarrow dx = \frac{1}{3} du$$

Grenzen: $u(0) = 1, u(1) = 4$

$$\begin{aligned} \dots &= \int_1^4 \sqrt{u} \cdot \frac{1}{3} du = \frac{1}{3} \int_1^4 \sqrt{u} du \\ &= \frac{1}{3} \left[\frac{2}{3} u^{3/2} \right]_1^4 = \frac{1}{3} \left[\frac{16}{3} - \frac{2}{3} \right] = \frac{14}{9} \end{aligned}$$

Aufgabe 3.2

$$\int x e^{x^2} dx$$

$$\text{Substitution: } u = x^2 \Rightarrow \frac{du}{dx} = 2x \Rightarrow dx = \frac{1}{2x} du$$

$$\begin{aligned} \dots &= \int x e^u \cdot \frac{1}{2x} du = \frac{1}{2} \int e^u du \\ &= \frac{1}{2} e^u = \frac{1}{2} e^{(x^2)} + C \end{aligned}$$

Aufgabe 3.3

$$\int \frac{4 \cdot \ln x}{x} dx$$

$$\text{Substitution: } u = \ln |x| \Rightarrow \frac{du}{dx} = \frac{1}{x} \Rightarrow dx = x \cdot du$$

$$\dots = 4 \int \frac{u}{x} \cdot x du = 4 \int u du = 4 \cdot \frac{1}{2} u^2 = 2(\ln |x|)^2 + C$$

Aufgabe 3.4

$$\int_{-2}^0 e^{-\frac{1}{2}x} dx$$

$$\text{Substitution: } u = -\frac{1}{2}x \Rightarrow \frac{du}{dx} = -\frac{1}{2} \Rightarrow dx = -2 \cdot du$$

Grenzen: $u(-2) = 1, u(0) = 0$

$$\begin{aligned} \dots &= \int_1^0 e^u \cdot (-2) \cdot du = (-2) \int_1^0 e^u du \\ &= (-2) [e^u]_1^0 = -2(1 - e) = 2e - 2 \end{aligned}$$

Aufgabe 3.5

$$\int \ln |5x| dx$$

Substitution: $u = 5x \Rightarrow \frac{du}{dx} = 5 \Rightarrow dx = \frac{1}{5} \cdot du$

$$\dots = \int \ln |u| \cdot \frac{1}{5} \cdot du = \frac{1}{5} u (\ln |u| - 1) = x (\ln |5x| - 1) + C$$

Aufgabe 3.6

$$\int (2x - 1)^3 dx$$

Substitution: $u = 2x - 1 \Rightarrow \frac{du}{dx} = 2 \Rightarrow dx = \frac{1}{2} du$

Grenzen: $u(1) = 1, u(2) = 3$

$$\dots = \int_1^3 u^3 \cdot \frac{1}{2} \cdot du = \frac{1}{8} [u^4]_1^3 = \frac{1}{8} (81 - 1) = 10$$

Aufgabe 3.7

$$\int x^2 \sin(x^3) dx$$

Substitution: $u = x^3 \Rightarrow \frac{du}{dx} = 3x^2 \Rightarrow dx = \frac{1}{3x^2} du$

$$\begin{aligned} \dots &= \int x^2 \sin(u) \cdot \frac{1}{3x^2} du = \frac{1}{3} \int \sin(u) du = -\frac{1}{3} \cos u + C_u \\ &= -\frac{1}{3} \cos(x^3) + C \end{aligned}$$

Aufgabe 3.8

$$\int_0^1 \frac{2x}{x^2 + 1} dx$$

Substitution: $u = x^2 + 1 \Rightarrow \frac{du}{dx} = 2x \Rightarrow dx = \frac{1}{2x} du$

Grenzen: $u(0) = 1, u(1) = 2$

$$\begin{aligned} \dots &= \int_1^2 \frac{2x}{u} \cdot \frac{1}{2x} du = \int_1^2 \frac{1}{u} du \\ &= [\ln |u|]_1^2 = \ln(2) - \ln(1) = \ln(2) \end{aligned}$$

Aufgabe 3.9

$$\int \frac{6}{(4-3x)^2} dx = \dots$$

$$u(x) = 4 - 3x \Rightarrow du = -3 dx \Rightarrow dx = -\frac{1}{3} du$$

$$\begin{aligned} \dots &= 6 \int \frac{1}{u^2} \cdot \frac{-1}{3} du = -2 \int \frac{1}{u^2} du \\ &= 2 \int u^{-1} + C_u = 2 \int (4-3x)^{-1} + C \end{aligned}$$

Aufgabe 3.10

$$\int_0^3 \frac{1}{\sqrt{t+1}} dt = \dots$$

$$u(x) = t + 1 \Rightarrow du = dx$$

Grenzen: $u(0) = 1, u(3) = 4$

$$\dots = \int_1^4 \frac{1}{\sqrt{u}} du = 2 \int_1^4 \frac{1}{2\sqrt{u}} du = 2[\sqrt{u}]_1^4 = 2(2-1) = 2$$

Aufgabe 3.11

$$\int \sqrt[3]{(3x-8)^2} dx = \dots$$

$$u(x) = 3x - 8 \Rightarrow du = 3 dx \Rightarrow dx = \frac{1}{3} du$$

$$\begin{aligned} \dots &= \frac{1}{3} \int \sqrt[3]{u^2} du = \frac{1}{3} \int u^{\frac{2}{3}} du \\ &= \frac{3}{3.5} u^{\frac{5}{3}} + C_u = \frac{1}{5} u^{\frac{5}{3}} + C_u = \frac{1}{5} (3x-8)^{\frac{5}{3}} + C \end{aligned}$$

Aufgabe 3.12

$$\int_0^1 \ln(x^2 + 1) dx = \dots$$

$$u(x) = x^2 + 1 \Rightarrow du = 2x dx \Rightarrow dx = \frac{1}{2x} du$$

Grenzen: $u(0) = 1, u(1) = 2$

$$\begin{aligned} \dots &= \int_1^2 x \ln|u| \cdot \frac{1}{2x} du = \frac{1}{2} \int_1^2 \ln|u| du \\ &= \frac{1}{2} [u(\ln|u| - 1)]_1^2 = \frac{1}{2} [2(\ln(2) - 1) - 1(\ln(1) - 1)] = \frac{1}{2}(2\ln(2) - 1) = \ln(2) - \frac{1}{2} \end{aligned}$$

Aufgabe 3.13

$$\int x(x^2 + 1)^2 dx = \dots$$

$$u(x) = x^2 + 1 \Rightarrow du = 2x dx \Rightarrow dx = \frac{1}{2x} du$$

$$\dots = \int x \cdot u^2 \cdot \frac{1}{2x} du = \frac{1}{2} \int u^2 du = \frac{1}{6}(x^2 + 1)^3 + C$$

Aufgabe 3.14

$$\int \frac{e^x}{e^x - 1} dx = \dots$$

$$u(x) = e^x - 1 \Rightarrow du = e^x dx \Rightarrow dx = e^{-x} du$$

$$\dots = \int \frac{e^x}{u} \cdot e^{-x} du = \int \frac{1}{u} du$$

$$= \ln |u| + C_u = \ln |e^x - 1| + C$$

Aufgabe 3.15

$$\int \frac{\ln(\sqrt{x} + 1)}{\sqrt{x}} dx = \dots$$

$$u(x) = \sqrt{x} + 1 \Rightarrow du = \frac{1}{2\sqrt{x}} dx \Rightarrow dx = 2\sqrt{x} du$$

$$\dots = \int \frac{\ln(u)}{\sqrt{x}} \cdot 2\sqrt{x} du = 2 \int \ln(u) du \\ = 2u(\ln|u| - 1) + C_u = 2(\sqrt{x} + 1)(\ln|\sqrt{x} + 1| - 1) + C$$

Aufgabe 3.16

$$\int_{\pi/2}^{\pi} \cos^2 x \sin x dx = \dots$$

$$u(x) = \cos x \Rightarrow du = -\sin x dx \Rightarrow dx = \frac{-1}{\sin x} du$$

$$\dots = \int u^2 \sin x \cdot \frac{-1}{\sin x} du = - \int u^2 du = -\frac{1}{3} \cos^3 x + C$$

$$\int_{\pi/2}^{\pi} \cos^2 x \sin x dx = \frac{1}{3} [\cos^3 x]_{\pi}^{\pi/2} = \frac{1}{3} (\cos^3 \frac{\pi}{2} - \cos^3 \pi) = \frac{1}{3}$$

Aufgabe 3.17 (★)

$$\int \frac{x}{1+x^4} dx = \dots$$

$$u(x) = x^2 \Rightarrow du = 2x \, dx \Rightarrow dx = \frac{1}{2x} \, du$$

$$\begin{aligned}\dots &= \int \frac{x}{1+u^2} \cdot \frac{1}{2x} \, du = \frac{1}{2} \int \frac{1}{1+u^2} \, du \\ &= \frac{1}{2} \arctan u + C_u = \frac{1}{2} \arctan x^2 + C\end{aligned}$$

Aufgabe 3.18 (*)

$$\int \frac{3+2t}{5+2t} \, dt = \dots$$

$$u(t) \stackrel{(*)}{=} 5+2t \Rightarrow du = 2 \, dt \Rightarrow dt = \frac{1}{2} \, du$$

$$\begin{aligned}\dots &= \frac{1}{2} \int \frac{3+2t}{u} \, du \stackrel{(*)}{=} \frac{1}{2} \int \frac{3+(u-5)}{u} \, du \\ &= \frac{1}{2} \int \frac{u-2}{u} \, du = \frac{1}{2} \int 1 \, du - \int \frac{1}{u} \, du \\ &= \frac{1}{2}u - \ln u + C_u = \frac{1}{2}(5+2t) - \ln(5+2t) + C\end{aligned}$$

Aufgabe 3.19

$$\int \frac{2x-1}{\sqrt{x^2-x-1}} \, dx = \dots$$

$$u = x^2 - x - 1 \Rightarrow du = (2x-1) \, dx \Rightarrow dx = \frac{1}{2x-1} \, du$$

$$\begin{aligned}\dots &= \int \frac{2x-1}{\sqrt{u}} \cdot \frac{1}{2x-1} \, du = \int u^{-\frac{1}{2}} \, du \\ &= 2u^{\frac{1}{2}} + C_u = 2\sqrt{x^2-x-1} + C\end{aligned}$$

Aufgabe 3.20

$$\int \frac{(\ln x)^2}{x} \, dx = \dots$$

$$u(x) = \ln x \Rightarrow du = x^{-1} \, dx \Rightarrow dx = x \, du$$

$$\begin{aligned}\dots &= \int \frac{u^2}{x} \cdot x \, du = \int u^2 \, du \\ &= \frac{1}{3}u^3 + C_u = \frac{1}{3}\ln^3 x + C\end{aligned}$$

Aufgabe 3.21 (*)

$$\int \sqrt{e^{3x} + e^{2x}} \, dx = \dots$$

$$x = \ln t \quad \Rightarrow \quad dx = t^{-1} dt$$

$$t = e^x$$

$$\begin{aligned} \dots &= \int \sqrt{e^{2x}(e^x + 1)} dx \\ &= \int e^x \sqrt{e^x + 1} dx = \int e^{\ln t} \sqrt{e^{\ln t} + 1} t^{-1} dt \\ &= \int t \sqrt{t+1} t^{-1} dt = \int (t+1)^{\frac{1}{2}} dt \\ &= \frac{2}{3}(t+1)^{\frac{3}{2}} + C_t = \frac{2}{3}(e^x + 1)^{\frac{3}{2}} + C \end{aligned}$$

Aufgabe 3.22 (*)

$$\int \frac{1}{1+\sqrt{x}} dx = \dots$$

$$x = t^2 \quad \Rightarrow \quad dx = 2t dt$$

$$t = \sqrt{x}$$

$$\dots = \int \frac{1}{1+\sqrt{t^2}} \cdot 2t dt = 2 \int \frac{t}{1+t} dt = \dots$$

$$u = 1+t \quad \Rightarrow \quad du = dt$$

$$\begin{aligned} \dots &= 2 \int \frac{u-1}{u} du = 2 \int 1 du - 2 \int \frac{1}{u} du = 2u - 2 \ln |u| + C_u \\ &= 2(1+t) - 2 \ln |1+t| + C_t \\ &= 2(1+\sqrt{x}) - 2 \ln |1+\sqrt{x}| + C \end{aligned}$$

Aufgabe 3.23 (*)

$$\int \frac{1+\ln x}{x(1-\ln x)} dx = \dots$$

$$x = e^t \quad \Rightarrow \quad dx = e^t dt$$

$$t = \ln x$$

$$\dots = \int \frac{1+\ln e^t}{e^t(1-\ln e^t)} e^t dt = \int \frac{1+t}{1-t} dt = \dots$$

$$u = 1-t \quad \Rightarrow \quad du = -dt \quad \Rightarrow \quad dt = -du$$

$$\begin{aligned} \dots &= - \int \frac{2-u}{u} du = \int \frac{u-2}{u} du = \int 1 du - 2 \int \frac{1}{u} du \\ &= u - 2 \ln |u| + C_u = 1 - t - 2 \ln |1-t| + C_t \\ &\stackrel{*}{=} -\ln x - 2 \ln |1-\ln x| + C \end{aligned}$$

(* Die 1 kann in die Integrationskonstante „integriert“ werden.)

Aufgabe 3.24

$$\int_0^\pi x \sin x \, dx = \dots$$

$$\begin{aligned} f'(x) &= \sin x &\Rightarrow f(x) &= -\cos x \\ g(x) &= x &\Rightarrow g'(x) &= 1 \end{aligned}$$

$$\begin{aligned} \dots &= [-x \cos x]_0^\pi + \int_0^\pi \cos x \cdot 1 \, dx \\ &= (-\pi \cos \pi - 0) + [\sin x]_0^\pi = -\pi \cdot (-1) + \sin \pi - \sin 0 = \pi \end{aligned}$$

Aufgabe 3.25

$$\int e^x \sin x \, dx = \dots$$

$$\begin{aligned} f'(x) &= e^x &\Rightarrow f(x) &= e^x \\ g(x) &= \sin x &\Rightarrow g'(x) &= \cos x \end{aligned}$$

$$\dots = e^x \sin x - \int e^x \cos x \, dx$$

$$\begin{aligned} f'(x) &= e^x &\Rightarrow f(x) &= e^x \\ g(x) &= \cos x &\Rightarrow g'(x) &= -\sin x \end{aligned}$$

$$\begin{aligned} \dots &= e^x \sin x - \left(e^x \cos x - \int e^x (-\sin x) \, dx \right) \\ &= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx \end{aligned}$$

Addiere auf beiden Seiten $\int e^x \sin x \, dx$:

$$2 \int e^x \sin x \, dx = e^x \sin x - e^x \cos x$$

$$\int e^x \sin x \, dx = \frac{1}{2} (e^x \sin x - e^x \cos x) + C$$

Aufgabe 3.26

$$\int \cos^2 x \, dx = \int \cos x \cos x \, dx = \dots$$

$$\begin{aligned} f'(x) &= \cos x &\Rightarrow f(x) &= \sin x \\ g(x) &= \cos x &\Rightarrow g'(x) &= -\sin x \end{aligned}$$

$$\begin{aligned}\dots &= \sin x \cos x - \int \sin x(-\sin x) dx \\ &= \sin x \cos x + \int \sin^2 x dx \\ &= \sin x \cos x + \int (1 - \cos^2 x) dx \\ &= \sin x \cos x + \int 1 dx - \int \cos^2 x dx\end{aligned}$$

Addiere auf beiden Seiten $\int \cos^2 x dx$:

$$\begin{aligned}2 \int \cos^2 x dx &= \sin x \cos x + x \\ \int \cos^2 x dx &= \frac{1}{2}(\sin x \cos x + x) + C\end{aligned}$$

Aufgabe 3.27

$$\int x^2 e^{-x} dx = \dots$$

$$\begin{aligned}f'(x) &= e^{-x} \Rightarrow f(x) = -e^{-x} \\ g(x) &= x^2 \Rightarrow g'(x) = 2x\end{aligned}$$

$$\dots = -x^2 e^{-x} + 2 \int x e^{-x} dx$$

$$\begin{aligned}f'(x) &= e^{-x} \Rightarrow f(x) = -e^{-x} \\ g(x) &= x \Rightarrow g'(x) = 1\end{aligned}$$

$$\begin{aligned}\dots &= -x^2 e^{-x} + 2 \left(-x e^{-x} + \int 1 e^{-x} dx \right) \\ &= -x^2 e^{-x} - 2x e^{-x} - e^{-x} + C \\ &= -(x^2 + 2x + 2)e^{-x} + C\end{aligned}$$

Aufgabe 3.28

$$\int x^3 \cdot e^x dx = e^x(x^3 - 3x^2 + 6x - 6) + C$$

Aufgabe 3.29

$$\int \sin x \cos x dx = \dots \text{ p}$$

$$\begin{aligned}f'(x) &= \cos x \Rightarrow f(x) = \sin x \\ g(x) &= \sin x \Rightarrow g'(x) = \cos x\end{aligned}$$

$$\int \sin x \cos x \, dx = \sin x \sin x - \int \cos x \sin x \, dx$$

Addiere auf beiden Seiten $\int \sin x \cos x \, dx$:

$$2 \int \sin x \cos x \, dx = \sin x \sin x$$

$$\int \sin x \cos x \, dx = \frac{1}{2} \sin^2 x + C$$

Aufgabe 3.30

$$\int x \ln |x| \, dx = \dots$$

$$f'(x) = x \quad \Rightarrow \quad f(x) = \frac{1}{2}x^2$$

$$g(x) = \ln |x| \quad \Rightarrow \quad g'(x) = x^{-1}$$

$$\dots = \frac{1}{2}x^2 \ln |x| - \frac{1}{2} \int x^2 \cdot x^{-1} \, dx$$

$$= \frac{1}{2}x^2 \ln |x| - \frac{1}{2} \int x \, dx$$

$$= \frac{1}{2}x^2 \ln |x| - \frac{1}{4}x^2 + C$$

$$\int_1^e x \ln |x| \, dx = \left[\frac{1}{2}x^2 \ln |x| - \frac{1}{4}x^2 \right]_1^e = \frac{1}{2}e^2 - \frac{1}{4}e^2 + \frac{1}{4} = \frac{1}{4}e^2 + \frac{1}{4}$$

Aufgabe 3.31

$$\int \ln^2 x \, dx = \int 1 \cdot \ln^2 x \, dx = \dots$$

$$f'(x) = 1 \quad \Rightarrow \quad f(x) = x$$

$$g(x) = \ln^2 x \quad \Rightarrow \quad g'(x) = 2 \ln x \cdot x^{-1}$$

$$\dots = x \ln^2 x - 2 \int x \ln x \cdot x^{-1} \, dx$$

$$= x \ln^2 x - 2 \int \ln x \, dx$$

$$= x \ln^2 x - 2x(\ln x - 1) + C$$

$$= x(\ln^2 x - 2 \ln x + 2) + C$$

Aufgabe 3.32

$$\int \frac{\ln x}{x^2} \, dx = \dots$$

$$f'(x) = x^{-2} \Rightarrow f(x) = -x^{-1}$$

$$g(x) = \ln x \Rightarrow g'(x) = x^{-1}$$

$$\dots = -x^{-1} \ln x + \int x^{-1} \cdot x^{-1} dx$$

$$= -x^{-1} \ln x + \int x^{-2} dx$$

$$= -x^{-1} \ln x - x^{-1} + C$$

Aufgabe 3.33

$$\int x^4 \ln x dx = \frac{1}{5}x^5 (\ln x - \frac{1}{5}) + C$$

Aufgabe 3.34

$$\begin{aligned} \int \sin^3 x dx &= -\frac{1}{3} \sin^2 x \cos x + \frac{2}{3} \int \sin x dx \\ &= -\frac{1}{3} \sin^2 x \cos x - \frac{2}{3} \cos x + C \end{aligned}$$

Aufgabe 3.35

$$\begin{aligned} \int \cos^4 x dx &= \frac{1}{4} \sin x \cos^3 x + \frac{3}{4} \int \cos^2 x dx \\ &= \frac{1}{4} \sin x \cos^3 x + \frac{3}{4} \cdot \frac{1}{2} (x + \sin x \cos x) + C \\ &= \frac{1}{4} \sin x \cos^3 x + \frac{3}{8} \sin x \cos x + \frac{3}{8} x + C \end{aligned}$$

Aufgabe 3.36

$$\begin{aligned} \int (2x^2 + 4x - 5)e^{2x} dx \\ &= e^{2x} \left[\frac{1}{2}(2x^2 + 4x - 5) - \frac{1}{4}(4x + 4) + \frac{1}{8} \cdot 4 \right] + C \\ &= e^{2x} \left[x^2 + 2x - \frac{5}{2} - x - 1 + \frac{1}{2} \right] + C \\ &= e^{2x} \left[x^2 + x - 3 \right] + C \end{aligned}$$

Aufgabe 3.37

$$x = \ln t$$

$$dx = \frac{1}{t} dt$$

Grenzen: damit x von 0 nach 1 läuft, muss t von 1 nach e laufen.

$$\begin{aligned}
\int_0^1 \frac{e^x}{1+e^x} dx &= \int_1^e \frac{e^{\ln t}}{1+e^{\ln t}} \cdot \frac{1}{t} dt = \int_1^e \frac{t}{1+t} \cdot \frac{1}{t} dt \\
&= \int_1^e \frac{1}{1+t} dt = [\ln|1+t|]_1^e = \ln(1+e) - \ln 2 \\
&= \ln \frac{1+e}{2} \approx 0.6201
\end{aligned}$$

Aufgabe 3.38

$$\int_0^1 (4x+1)^3 dx = \dots$$

Substitution: $u = 4x + 1 \Rightarrow \frac{du}{dx} = 4 \Rightarrow dx = \frac{1}{4}du$

$$\dots = \int_{u(0)}^{u(1)} u^3 \cdot \frac{1}{4} du = \frac{1}{4} \int_1^5 u^3 du = \frac{1}{16} [u^4]_1^5 = \frac{1}{16} (625 - 1) = 39$$

Aufgabe 3.39

$$\int_0^1 xe^{-x} dx = \dots$$

$$f'(x) = e^{-x} \Rightarrow f(x) = -e^{-x}$$

$$g(x) = x \Rightarrow g'(x) = 1$$

$$\begin{aligned}
\dots &= [x \cdot (-e^{-x})]_0^1 - \int_0^1 1 \cdot (-e^{-x}) dx = [-x \cdot e^{-x}]_0^1 + \int_0^1 e^{-x} dx \\
&= [x \cdot (-e^{-x})]_0^1 - \int_0^1 1 \cdot (-e^{-x}) dx \\
&= [-x \cdot e^{-x}]_0^1 + [-e^{-x}]_0^1 \\
&= (-1 \cdot e^{-1} + 0 \cdot e^0) + (-e^{-1} + e^0) = 1 - 2e^{-1}
\end{aligned}$$

Aufgabe 3.40

$$\int_0^1 \frac{3x}{x^2 + 9} dx = \dots$$

Substitution: $u = x^2 + 9 \Rightarrow \frac{du}{dx} = 2x \Rightarrow dx = \frac{1}{2x}du$
--

$$\begin{aligned}
\dots &= \int_{u(0)}^{u(1)} \frac{3x}{u} \cdot \frac{1}{2x} du = \frac{3}{2} \int_9^{10} \frac{1}{u} du = \frac{3}{2} \cdot [\ln|u|]_9^{10} \\
&= \frac{3}{2} (\ln(10) - \ln(9)) = \frac{3}{2} \ln \left(\frac{10}{9} \right)
\end{aligned}$$

Aufgabe 3.41

$$\int_{-1}^1 x^2 \cdot e^{(x^3)} dx = \dots$$

Substitution: $u = x^3 \Rightarrow \frac{du}{dx} = 3x^2 \Rightarrow dx = \frac{1}{3x^2} du$

$$\dots = \int_{u(-1)}^{u(1)} x^2 e^u \cdot \frac{1}{3x^2} du = \frac{1}{3} \int_{-1}^1 e^u du = \frac{1}{3} [e^u]_{-1}^1 = \frac{1}{3} (e - e^{-1})$$

Aufgabe 3.42

$$\int_0^{\pi/2} \sin^2 x \cdot \cos x dx = \dots$$

$$\text{Substitution: } u = \sin x \Rightarrow \frac{du}{dx} = \cos x \Rightarrow dx = \frac{1}{\cos x} du$$

Grenzen: $u(0) = 0, u(\pi/2) = 1$

$$\dots = \int_0^1 u^2 \cdot \cos x \cdot \frac{1}{\cos x} du = \int_0^1 u^2 du = \frac{1}{3} [u^3]_0^1 = \frac{1}{3} (1 - 0) = \frac{1}{3}$$

Aufgabe 3.43

$$\int_1^2 x^2 \ln x dx = \dots$$

$$f'(x) = x^2 \Rightarrow f(x) = \frac{1}{3}x^3$$

$$g(x) = \ln(x) \Rightarrow g'(x) = \frac{1}{x}$$

$$\dots = \left[\frac{1}{3}x^3 \cdot \ln x \right]_1^2 - \int_1^2 \frac{1}{3}x^3 \cdot \frac{1}{x} dx$$

$$= \left[\frac{1}{3}x^3 \cdot \ln x \right]_1^2 - \frac{1}{3} \int_1^2 x^2 dx$$

$$= \left[\frac{1}{3}x^3 \cdot \ln x \right]_1^2 - \left[\frac{1}{9}x^3 \right]_1^2$$

$$= \left(\frac{8}{3} \cdot \ln 2 - \frac{1}{3} \cdot \ln 1 \right) - \left(\frac{8}{9} - \frac{1}{9} \right)$$

$$= \frac{8}{3} \cdot \ln 2 - \frac{1}{3} \cdot 0 - \frac{7}{9}$$

$$= \frac{8}{3} \cdot \ln 2 - \frac{7}{9}$$

Aufgabe 3.44

$$\int_0^1 x(x^2 + 1)^3 dx = \dots$$

$$\text{Substitution } u(x) = x^2 + 1 \Rightarrow \frac{du}{dx} = 2x \Rightarrow dx = \frac{1}{2x} du$$

Die Transformation der Grenzen werden hier an Ort und Stelle vorgenommen:

$$\dots = \int_{u(0)}^{u(1)} x \cdot u^3 \cdot \frac{1}{2x} du = \int_1^2 \frac{1}{2} u^3 du = \frac{1}{8} [u^4]_1^2 = \frac{1}{8} (16 - 1) = \frac{15}{8}$$

Aufgabe 3.45

$$\int_1^e \frac{\sqrt{\ln x}}{x} dx = \dots$$

$$\text{Substitution: } u(x) = \ln(x) \Rightarrow \frac{du}{dx} = \frac{1}{x} \Rightarrow dx = x du$$

Die Grenzen werden wieder vor Ort substituiert:

$$\dots = \int_{u(1)}^{u(e)} \frac{\sqrt{u}}{x} \cdot x du = \int_0^1 \sqrt{u} du = \left[\frac{2}{3} u^{3/2} \right]_0^1 = \frac{2}{3}$$

Aufgabe 3.46

$$\int 1 \cdot \arccos x dx = \dots$$

$$f'(x) = 1 \Rightarrow f(x) = x$$

$$g(x) = \arccos x \Rightarrow g'(x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\dots = x \arccos x - \int x \cdot \frac{-1}{\sqrt{1-x^2}} dx = \dots$$

$$u(x) = 1 - x^2 \Rightarrow \frac{du}{dx} = -2x \Rightarrow dx = -\frac{1}{2x} du$$

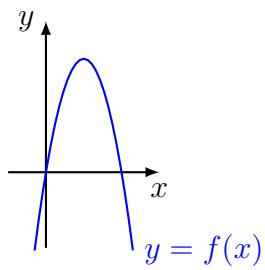
$$\dots = x \arccos x - \int x \cdot \frac{1}{u} \cdot \frac{-1}{2x} du$$

$$= x \arccos x + \int \frac{1}{2\sqrt{u}} du$$

$$= x \arccos x + \sqrt{u} + C = x \arccos x + \sqrt{1-x^2} + C$$

Aufgabe 4.1

Skizze:



Nullstellen:

$$6x - 3x^2 = 0$$

$$3x(2 - x) = 0$$

$$x_1 = 0$$

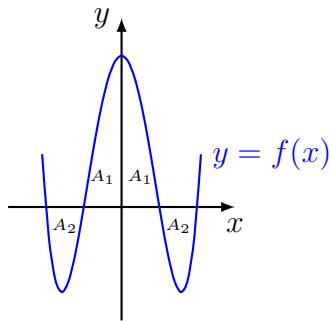
$$x_2 = 2$$

$$A = \int_0^2 (6x - 3x^2) dx = [3x^2 - x^3]_0^2$$

$$= (12 - 8) - (0 - 0) = 4 \text{ FE}$$

Aufgabe 4.2

Skizze: (Ordinatensymmetrie)



Nullstellen:

$$\begin{aligned}
 x^4 - 5x^2 + 4 &= 0 \\
 (x^2 - 4)(x^2 - 1) &= 0 \\
 (x - 2)(x + 2)(x - 1)(x + 1) &= 0 \\
 x_1 &= -2 \\
 x_2 &= -1 \\
 x_3 &= 1 \\
 x_4 &= 2
 \end{aligned}$$

$$\begin{aligned}
 A_1 &= \int_0^1 (x^4 - 5x^2 + 4) dx = \left[\frac{1}{5}x^5 - \frac{5}{3}x^3 + 4x \right]_0^1 \\
 &= \left(\frac{1}{5} - \frac{5}{3} + 4 \right) - (0 - 0 + 0) = \frac{3}{15} - \frac{25}{15} + \frac{60}{15} = \frac{38}{15}
 \end{aligned}$$

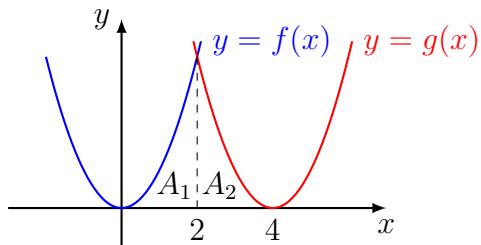
$$\begin{aligned}
 A_2 &= \int_{-2}^2 (x^4 - 5x^2 + 4) dx = \left[\frac{1}{5}x^5 - \frac{5}{3}x^3 + 4x \right]_{-2}^2 \\
 &= \left(\frac{1}{5} - \frac{5}{3} + 4 \right) - \left(\frac{32}{5} - \frac{40}{3} + 8 \right) \\
 &= -\frac{31}{5} + \frac{35}{3} - 4 = -\frac{93}{15} + \frac{175}{15} - \frac{60}{15} = \frac{22}{15}
 \end{aligned}$$

Aufgrund der Symmetrie gilt:

$$A_{\text{Total}} = 2 \cdot (A_1 + A_2) = 2 \cdot \left(\frac{38}{15} + \frac{22}{15} \right) = 2 \cdot \frac{60}{15} = 2 \cdot 4 = 8 \text{ FE}$$

Aufgabe 4.3

Skizze:



Schnittstelle(n):

$$x^2 = (x - 4)^2$$

$$x^2 = x^2 - 8x + 16$$

$$0 = -8x + 16$$

$$x = 2$$

Nullstelle von g :

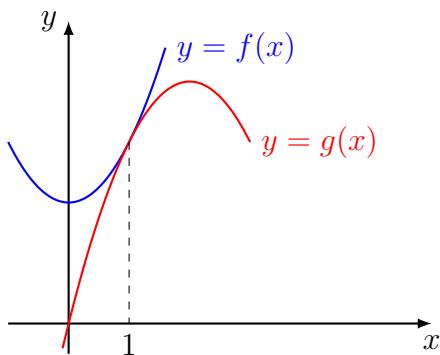
$$(x - 4)^2 = 0 \Rightarrow x = 4$$

$$A_1 = \int_0^2 x^2 dx = \left[\frac{1}{3}x^3 \right]_0^2 = \frac{8}{3} - 0 = \frac{8}{3}$$

Aufgrund der Symmetrie gilt: $A_1 = A_2$ und damit: $A = 2 \cdot A_1 = \frac{16}{3}$ FE

Aufgabe 4.4

Skizze:



Berührstelle:

$$2 + x^2 = 4x - x^2$$

$$2x^2 - 4x + 2 = 0$$

$$x^2 - 2x + 1 = 0$$

$$(x - 1)^2 = 0$$

$$x = 1$$

Nachweis der Berührungs: (gleiche Steigung bei $x = 1$)

$$f'(x) = 2x \Rightarrow f'(1) = 2$$

$$g'(x) = 4 - 2x \Rightarrow g'(1) = 2$$

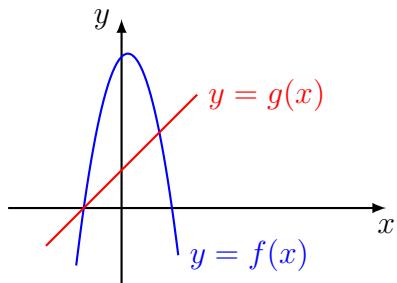
$$A_f = \int_0^1 (2 + x^2) dx = \left[2x + \frac{1}{3}x^3 \right]_0^1 = 2 + \frac{1}{3} = \frac{7}{3}$$

$$A_g = \int_0^1 (4x - x^2) dx = \left[2x^2 - \frac{1}{3}x^3 \right]_0^1 = 2 - \frac{1}{3} = \frac{5}{3}$$

$$A_{\text{Total}} = A_f - A_g = \frac{7}{3} - \frac{5}{3} = \frac{2}{3} \text{ FE}$$

Aufgabe 4.5

Skizze:



$$\text{Schnitstellen: } -3x^2 + x + 4 = x + 1$$

$$3 = 3x^2$$

$$x^2 = 1$$

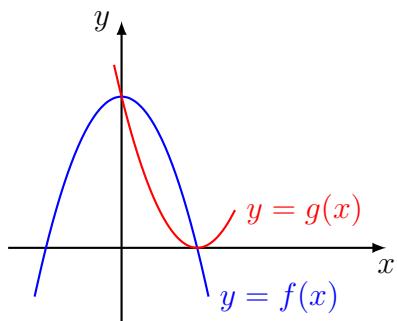
$$x_1 = -1$$

$$x_2 = 1$$

$$\begin{aligned} A &= \int_{-1}^1 [f(x) - g(x)] dx = \int_{-1}^1 (-3x^2 + x + 4 - (x + 1)) dx \\ &= \int_{-1}^1 (-3x^2 + x + 4 - x - 1) dx \\ &= \int_{-1}^1 (-3x^2 + 3) dx = [-x^3 + 3x]_{-1}^1 \\ &= (-1 + 3) - (1 - 3) = 4 \text{ FE} \end{aligned}$$

Aufgabe 4.6

Skizze:



$$\text{Schnitstellen: } 4 - x^2 = x^2 - 4x + 4$$

$$0 = 2x^2 - 4x = 2x(x - 2)$$

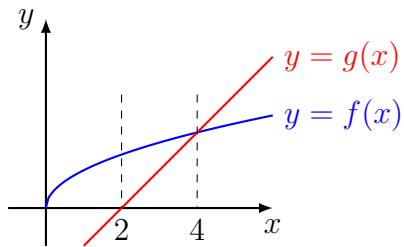
$$x_1 = 0$$

$$x_2 = 2$$

$$\begin{aligned} A &= \int_0^2 [f(x) - g(x)] dx = \int_0^2 (4 - x^2 - x^2 + 4x - 4) dx \\ &= \int_0^2 (-2x^2 + 4x) dx = \left[-\frac{2}{3}x^3 + 2x^2 \right]_0^2 \\ &= \left(-\frac{16}{3} + 8 \right) - (0 + 0) = \frac{8}{3} \text{ FE} \end{aligned}$$

Aufgabe 4.7

Skizze:



Schnittpunkte:

$$\sqrt{x} = x - 2 \quad ||^2$$

$$x = (x - 2)^2 = x^2 - 4x + 4 \quad || - x$$

$$0 = x^2 - 5x + 4$$

$$0 = (x - 1)(x - 4)$$

$$x_1 = -1 \quad (\text{Probe: } 1 = -1 \Rightarrow \text{Scheinlösung})$$

$$x_2 = 4 \quad (\text{Probe: } 2 = 2 \Rightarrow \text{ok})$$

Nullstelle von g : $x - 2 = 0$

$$x_1 = 2$$

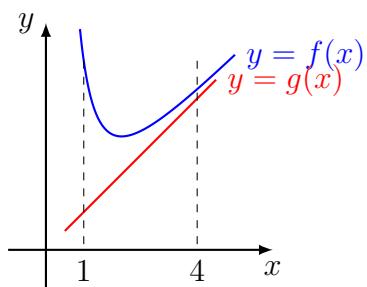
$$\begin{aligned} A_1 &= \int_0^4 \sqrt{x} dx = \frac{2}{3} [x^{3/2}]_0^4 = \frac{2}{3} [\sqrt{x^3}]_0^4 \\ &= \frac{2}{3} (\sqrt{64} - \sqrt{0}) = \frac{16}{3} \end{aligned}$$

$$A_2 = \int_2^4 (x - 2) dx = \left[\frac{1}{2}x^2 - 2x \right]_2^4 = (8 - 8) - (2 - 4) = 2$$

$$A = A_1 - A_2 = \frac{16}{3} - 2 = \frac{10}{3} \text{ FE}$$

Aufgabe 4.8

Skizze:



Gleichung der schießen Asymptote:

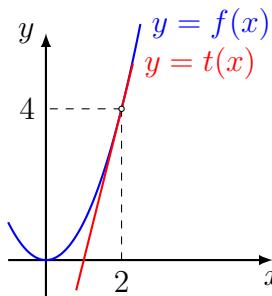
$$f(x) = x + 4/x^2 \approx x \text{ für grosse } |x| \quad \Rightarrow \quad g(x) = x$$

Schnittstellen: keine

$$\begin{aligned} A &= \int_1^4 (f(x) - g(x)) dx = \int_1^4 (x + 4x^{-2} + x) dx = \int_1^4 4x^{-2} dx \\ &= [-4x^{-1}]_1^4 = -\frac{4}{4} - \left(-\frac{4}{1}\right) = 3 \text{ FE} \end{aligned}$$

Aufgabe 4.9

Skizze:



Gleichung der Tangente im Punkt $P(2, 4)$:

$$f'(x) = 2x \Rightarrow f'(2) = 2 \cdot 2 = 4 = m \quad (\text{Steigung})$$

$$y = mx + q$$

$$4 = 4 \cdot 2 + q$$

$$q = -4 \quad (\text{Ordinatenabschnitt})$$

$$\Rightarrow t: y = 4x - 4$$

$$\text{Nullstelle von } f: x^2 = 0 \Rightarrow x = 0$$

$$\text{Nullstelle von } t: 0 = 4x - 4 \Rightarrow x = 1$$

$$A_1 = \int_0^2 x^2 dx = \left[\frac{1}{3}x^3 \right]_0^2 = \frac{8}{3}$$

$$A_2 = \frac{1 \cdot 4}{2} = 2 \quad (\text{Dreiecksfläche; Integrieren geht auch})$$

$$A = A_1 - A_2 = \frac{8}{3} - 2 = \frac{2}{3} \text{ FE}$$

Aufgabe 4.10

Setzt man die Koordinaten des Punkts $(2, 2)$ in die Gleichung $y = ax^2$ ein, so erhält man $2 = a \cdot 4$ und daraus $a = \frac{1}{2}$.

Inhalt der Fläche *unter* dem Graphen:

$$\int_0^2 \frac{1}{2}x^2 dx = \left[\frac{1}{6}x^3 \right]_0^2 = \frac{8}{6} = \frac{4}{3} \text{ FE}$$

$$\text{Gesuchter Flächeninhalt: } A = 4 - \frac{4}{3} = \frac{8}{3} \text{ FE}$$

Aufgabe 4.11

$$A = \int_0^2 f(x) dx = \left[\frac{1}{6}x^3 + \frac{1}{6}x^2 + x \right]_0^2 = 4$$

$$\frac{A}{2} = \int_0^m f(x) dx$$

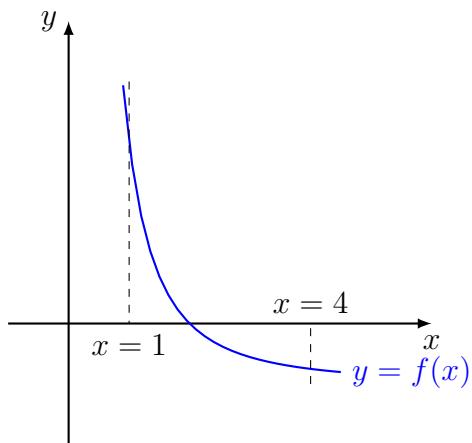
$$2 = \left[\frac{1}{6}x^3 + \frac{1}{6}x^2 + x \right]_0^m$$

$$2 = \frac{1}{6}m^3 + \frac{1}{6}m^2 + m$$

$$0 = \frac{1}{6}m^3 + \frac{1}{6}m^2 + m - 2 \Rightarrow m = 1.32$$

Aufgabe 4.12

Skizze:



Nullstelle:

$$\frac{4}{x^2} - 1 = 0 \quad || \cdot x^2 \neq 0$$

$$4 - x^2 = 0$$

$$x_1 = -2 \quad (\text{liegt nicht im Intervall})$$

$$x_2 = 2$$

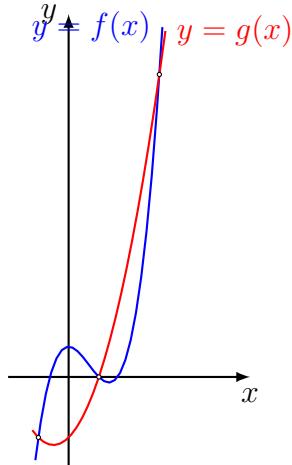
$$\begin{aligned} A_1 &= \int_1^2 (4x^{-2} - 1) dx = [-4x^{-1} - x]_1^2 \\ &= (-2 - 2) - (-4 - 1) = -4 + 5 = 1 \end{aligned}$$

$$\begin{aligned} A_2 &= \int_4^2 (4x^{-2} - 1) dx = [-4x^{-1} - x]_4^2 \\ &= (-2 - 2) - (-1 - 4) = -4 + 5 = 1 \end{aligned}$$

$$A = A_1 + A_2 = 1 + 1 = 2 \text{ FE}$$

Aufgabe 4.13

Skizze:



Schnittpunkte:

$$\begin{aligned} x^3 - 2x^2 + 1 &= x^2 + x - 2 \quad || - x^2 - x + 2 \\ x^3 - 3x^2 - x + 3 &= 0 \end{aligned}$$

$$x_1 = -1$$

$$x_2 = 1$$

$$x_3 = 3$$

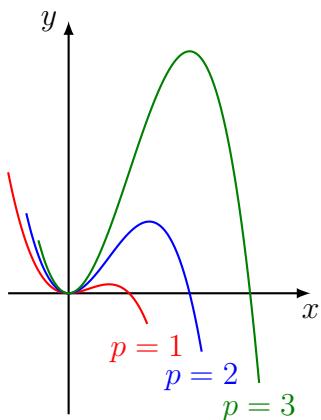
$$\begin{aligned} A_1 &= \int_{-1}^1 (x^3 - 2x^2 + 1 - (x^2 + x - 2)) dx \\ &= \int_{-1}^1 (x^3 - 3x^2 - x + 3) dx = \left[\frac{1}{4}x^4 - x^3 - \frac{1}{2}x^2 + 3x \right]_{-1}^1 \\ &= \left(\frac{1}{4} - 1 - \frac{1}{2} + 3 \right) - \left(\frac{1}{4} + 1 - \frac{1}{2} - 3 \right) = -2 + 6 = 4 \end{aligned}$$

$$\begin{aligned} A_2 &= \int_3^1 [f(x) - g(x)] dx \\ &= \int_3^1 (x^3 - 3x^2 - x + 3) dx = \left[\frac{1}{4}x^4 - x^3 - \frac{1}{2}x^2 + 3x \right]_3^1 \\ &= \left(\frac{1}{4} - 1 - \frac{1}{2} + 3 \right) - \left(\frac{81}{4} - 27 - \frac{9}{2} + 9 \right) = 4 \end{aligned}$$

$$A = A_1 + A_2 = 4 + 4 = 8 \text{ FE}$$

Aufgabe 4.14

Skizze:



Offenbar haben alle Funktionen bei $x = 0$ eine doppelte Nullstelle und eine Nullstelle bei $x = p$, was durch folgende Rechnung zur Nullstellenbestimmung bestätigt wird:

$$px^2 - x^3 = 0$$

$$x^2(p - x) = 0$$

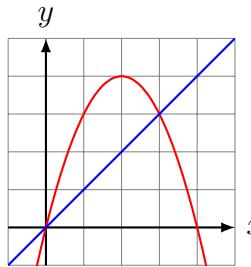
$$x_1 = x_2 = 0$$

$$x_3 = p$$

Mit dem gegebenen Flächeninhalt ($A = 4/3$) lässt sich folgende Gleichung aufstellen:

$$\begin{aligned} \int_0^p (px^2 - x^3) dx &= \frac{4}{3} \\ \left[p \cdot \frac{1}{3}x^3 - \frac{1}{4}x^4 \right]_0^p &= \frac{4}{3} \\ \left(p \cdot \frac{1}{3}p^3 - \frac{1}{4}p^4 \right) - (0 - 0) &= \frac{4}{3} \\ \frac{1}{3}p^4 - \frac{1}{4}p^4 &= \frac{4}{3} \quad || \cdot 12 \\ 4p^4 - 3p^4 &= 16 \\ p^4 &= 16 \\ p &= 2 \quad (p > 0) \end{aligned}$$

Aufgabe 4.15



$$G_f \cap G_g: \quad 4x - x^2 = mx$$

$$(4 - m)x - x^2 = 0$$

$$x(4 - m - x) = 0$$

$$x_1 = 0$$

$$x_2 = 4 - m$$

$$A = \int_0^4 (4x - x^2) dx = [2x^2 - \frac{1}{3}x^3]_0^4 = (32 - \frac{64}{3}) = \frac{32}{3}$$

$$\int_0^{4-m} (4x - x^2 - mx) dx = \frac{16}{3}$$

$$[2x^2 - \frac{1}{3}x^3 - \frac{m}{2}x^2]_0^{4-m} = \frac{16}{3} \quad || \cdot 6$$

$$[12x^2 - 2x^3 - 3mx^2]_0^{4-m} = 32$$

$$12(4 - m)^2 - 2(4 - m)^3 - 3m(4 - m)^2 = 32$$

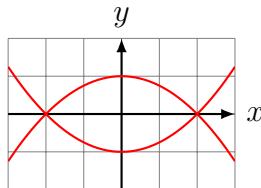
$$(4 - m)^2 [12 - 2(4 - m) - 3m] = 32$$

$$(4 - m)^2 [4 - m] = 32$$

$$(4 - m)^3 = 32$$

$$a = 4 - \sqrt[3]{32} \approx 0.8252$$

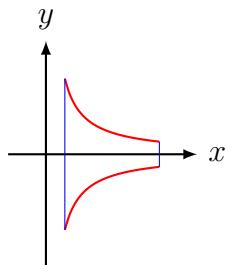
Aufgabe 5.1



$$\text{Nullstellen: } \frac{1}{4}(x^2 - 4) = 0 \quad \Rightarrow \quad x = \pm 2$$

$$\begin{aligned} V &= 2\pi \int_0^2 \left[\frac{1}{4}(x^2 - 4) \right]^2 dx \\ &= \frac{1}{8}\pi \int_0^2 (x^4 - 4x^2 + 16) dx \\ &= \frac{1}{8}\pi \left[\frac{1}{5}x^5 - \frac{4}{3}x^3 + 16x \right]_0^2 = \frac{32}{15}\pi \end{aligned}$$

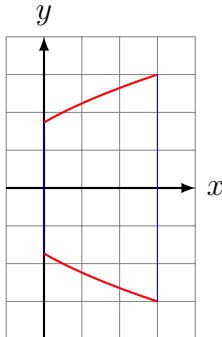
Aufgabe 5.2



$$\begin{aligned} V &= \pi \int_a^b (x^{-1})^2 dx = \pi \int_a^b (x^{-2}) dx = \pi [-x^{-1}]_a^b \\ &= \pi \left(-\frac{1}{b} + \frac{1}{a} \right) = \pi \left(\frac{1}{a} - \frac{1}{b} \right) \end{aligned}$$

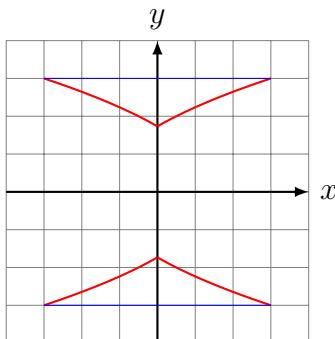
Aufgabe 5.3

$$(a) \ y^2 = 2x + 3 \Leftrightarrow y = \pm\sqrt{2x+3}$$



$$\begin{aligned} V_x &= \pi \int_1^3 (2x+3) dx = \pi \int_1^3 (x^2 + 2x + 1) dx \\ &= \pi [x^2 + 2x + 1]_1^3 = 18\pi \end{aligned}$$

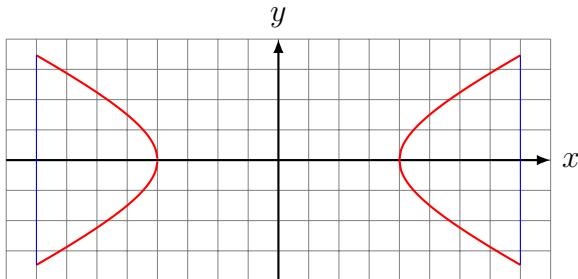
$$(b) \ y^2 = 2x + 3 \Leftrightarrow x = \frac{1}{2}(y^2 - 3)$$



$$\begin{aligned} V_y &= 2\pi \int_{\sqrt{3}}^3 \frac{1}{4}(y^2 - 3)^2 dy = \frac{\pi}{2} \int_{\sqrt{3}}^3 (y^4 - 6y^2 + 9) dy \\ &= \frac{\pi}{2} \left[\frac{1}{5}y^5 - 2y^3 + 9y \right]_{\sqrt{3}}^3 \\ &= \frac{\pi}{2} \left[\left(\frac{243}{5} - 54 + 27 \right) - \left(\frac{9\sqrt{3}}{5} - 6\sqrt{3} + 9\sqrt{3} \right) \right] \\ &= \frac{\pi}{2} \left[\frac{108}{5} - \frac{24\sqrt{3}}{5} \right] \\ &= \frac{\pi}{5} (54 - 12\sqrt{3}) = \frac{6\pi}{5} (9 - 2\sqrt{3}) \end{aligned}$$

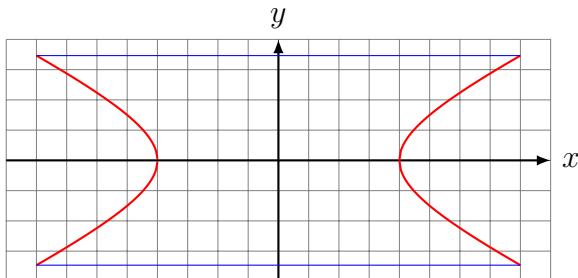
Aufgabe 5.4

$$(a) \quad x^2 - 4y^2 = 16 \quad \Leftrightarrow \quad y^2 = \frac{1}{4}(x^2 - 16) \quad (4 \leq x \leq 8)$$



$$\begin{aligned} V_x &= 2\pi \int_4^8 \frac{1}{4}(x^2 - 16) dx = \frac{\pi}{2} \left[\frac{1}{3}x^3 - 16x \right]_4^8 \\ &= \frac{\pi}{2} \left[\left(\frac{512}{3} - 128 \right) - \left(\frac{64}{3} - 64 \right) \right] = \frac{\pi}{2} \cdot \frac{256}{3}\pi = \frac{128}{3}\pi \end{aligned}$$

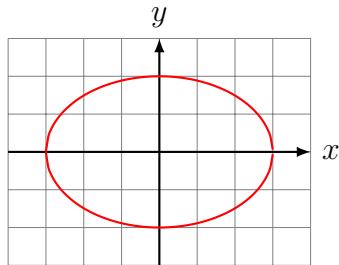
$$(b) \quad x^2 - 4y^2 = 16 \quad \Leftrightarrow \quad x^2 = \sqrt{16 + 4y^2}$$



$$\begin{aligned} V_x &= 2\pi \int_0^{2\sqrt{3}} (16 + 4y^2) dy = 2\pi \left[16y + \frac{4}{3}y^3 \right]_0^{2\sqrt{3}} \\ &= 2\pi \left[32\sqrt{3} + 32\sqrt{3} \right] = 128\sqrt{3}\pi \end{aligned}$$

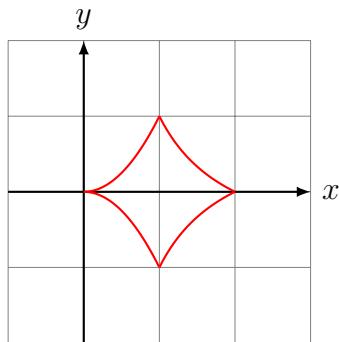
Aufgabe 5.5

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \Rightarrow \quad y^2 = b^2 - \frac{b^2}{a^2}x^2$$



$$\begin{aligned} V_x &= 2\pi \int_0^a \left(b^2 - \frac{b^2}{a^2}x^2 \right) dx = 2b^2\pi \left[x - \frac{1}{3a^2}x^3 \right]_0^a \\ &= 2b^2\pi \left[a - \frac{1}{3}a \right] = \frac{4}{3}ab^2\pi \end{aligned}$$

Aufgabe 5.6



Schnittstelle:

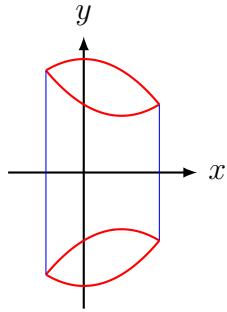
$$\begin{aligned} x^2 &= 2/x - 1 \\ x^3 &= 2 - x \\ x^3 + x - 2 &= 0 \\ x &= 1 \quad (\text{einzige reelle Nullstelle}) \end{aligned}$$

$$V_1 = \pi \int_0^1 (x^2)^2 dx = \pi \int_0^1 x^4 dx = \pi \left[\frac{1}{5}x^5 \right]_0^1 = \frac{1}{5}\pi$$

$$\begin{aligned} V_2 &= \pi \int_1^2 (2x^{-1} - 1)^2 dx = \pi \int_1^2 (4x^{-2} - 4x^{-1} + 1) dx \\ &= \pi \left[-4x^{-1} - 4\ln(x) + x \right]_1^2 \\ &= \pi [(-2 - 4\ln(2) + 2) - (-4 - 0 + 1)] \\ &= \pi [3 - 4\ln(2)] \end{aligned}$$

$$V = V_1 + V_2 = \pi [3.2 - 4\ln(2)]$$

Aufgabe 5.7



Schnittstellen: $f(x) = g(x)$

$$x^2 - 2x + 6 = -x^2 + 10$$

$$2x^2 - 2x - 4 = 0$$

$$x^2 - x - 2 = 0$$

$$x_1 = -1$$

$$x_2 = 2$$

Äusseres Volumen:

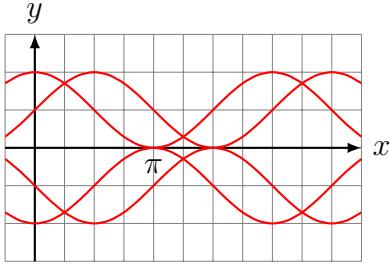
$$\begin{aligned} V_1 &= \pi \int_{-1}^2 (10 - x^2)^2 dx \\ &= \pi \int_{-1}^2 (100 - 20x^2 + x^4) dx \\ &= \pi \left[100x - \frac{20}{3}x^3 + \frac{1}{5}x^5 \right]_{-1}^2 = \frac{1233}{5}\pi \end{aligned}$$

Inneres Volumen:

$$\begin{aligned} V_2 &= \pi \int_{-1}^2 (x^2 - 2x + 6)^2 dx \\ &= \pi \int_{-1}^2 (x^4 + 4x^2 + 36 - 4x^3 - 24x + 12x^2) dx \\ &= \pi \int_{-1}^2 (x^4 - 4x^3 + 16x^2 - 24x + 36) dx \\ &= \pi \left[\frac{1}{5}x^5 - x^4 + \frac{16}{3}x^3 - 12x^2 + 36x \right]_{-1}^2 = \frac{558}{5}\pi \end{aligned}$$

$$\text{Differenz: } V = V_2 - V_1 = \frac{1233}{5}\pi - \frac{558}{5}\pi = \frac{675}{5}\pi = 135\pi$$

Aufgabe 5.8



Schnitstellen: $1 + \sin(x) = 1 + \cos(x)$

$$\sin(x) = \cos(x)$$

$$\frac{\sin(x)}{\cos(x)} = 1$$

$$\tan(x) = 1$$

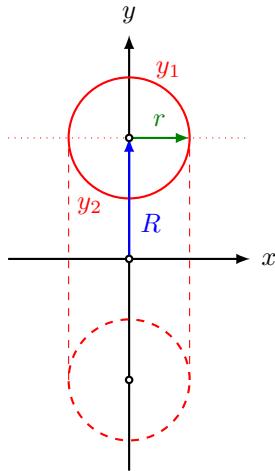
$$x_k = \frac{\pi}{4} + k\pi$$

$$\begin{aligned} V_2 &= \pi \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (1 + \sin x)^2 dx \\ &= \pi \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (1 + 2 \sin x + \sin^2 x) dx \\ &= \pi \left[x - 2 \cos x + \frac{1}{2}x - \frac{1}{4} \sin 2x \right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \\ &= \pi \left[\frac{3}{2}x - 2 \cos x - \frac{1}{4} \sin 2x \right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \\ &= \pi \left[\frac{3}{2} + 2\sqrt{2} \right] \end{aligned}$$

$$\begin{aligned} V_1 &= \pi \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (1 + \cos x)^2 dx \\ &= \pi \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (1 + 2 \cos x + \cos^2 x) dx \\ &= \pi \left[x - 2 \cos x + \frac{1}{2}x - \frac{1}{4} \sin(2x) \right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \\ &= \pi \left[\frac{3}{2} - 2\sqrt{2} \right] \end{aligned}$$

$$V = V_2 - V_1 = 4\sqrt{2}\pi$$

Aufgabe 5.9



$$y_1 = R + \sqrt{r^2 - x^2} \quad \text{oberer Halbkreis}$$

$$y_2 = R - \sqrt{r^2 - x^2} \quad \text{unterer Halbkreis}$$

$$V_1 = 2\pi \int_0^r (R + \sqrt{r^2 - x^2})^2 dx$$

$$V_2 = 2\pi \int_0^r (R - \sqrt{r^2 - x^2})^2 dx$$

$$V = V_1 - V_2$$

$$= 2\pi \int_0^r [(R^2 + 2R\sqrt{*} + \sqrt{*}^2) - (R^2 - 2R\sqrt{*} + \sqrt{*}^2)] dx$$

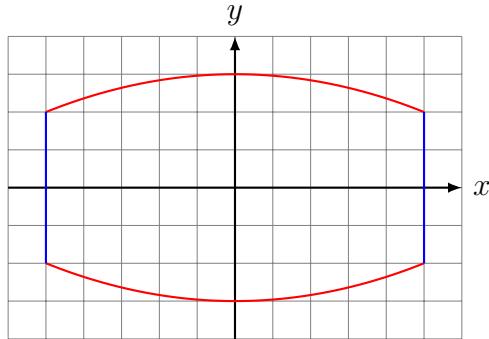
$$= 8\pi R \int_0^r \sqrt{r^2 - x^2} dx$$

$$= 8\pi R \left[\frac{x}{2} \sqrt{r^2 - x^2} + \frac{r^2}{2} \arcsin \frac{x}{r} \right]_0^r$$

$$= 8\pi R \left[\left(0 + \frac{r^2}{2} \arcsin 1 \right) - \left(0 + \frac{r^2}{2} \arcsin 0 \right) \right]$$

$$= 8\pi R \cdot \frac{r^2}{2} \cdot \frac{\pi}{2} = 2\pi^2 R r^2$$

Aufgabe 5.10



$$f(x) = ax^2 + b$$

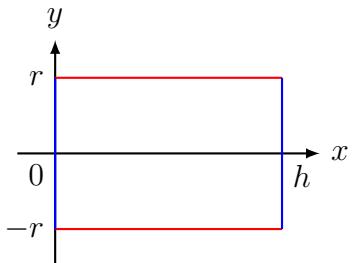
$$f(0) = 3 = b$$

$$f(5) = 2 = 25a + 3 \Rightarrow a = -\frac{1}{25}$$

$$f(x) = 3 - \frac{1}{25}x^2$$

$$\begin{aligned} V &= 2\pi \int_0^5 \left(3 - \frac{1}{25}x^2\right)^2 dx \\ &= 2\pi \int_0^5 \left(9 - \frac{6}{25}x^2 + \frac{1}{625}x^4\right) dx \\ &= 2\pi \left[9x - \frac{2}{25}x^3 + \frac{1}{3125}x^5\right]_0^5 \\ &= 2\pi[45 - 10 + 1] = 72\pi \text{ dm}^3 \approx 226.19 \text{ Liter} \end{aligned}$$

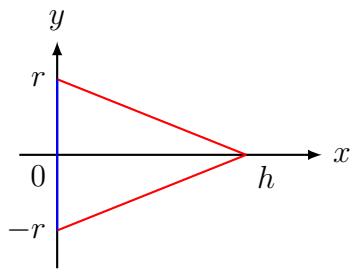
Aufgabe 5.11



$$f(x) = r \Rightarrow f'(x) = 0$$

$$M = 2\pi \int_0^h r\sqrt{1+0^2} dx = 2\pi \int_0^h r dx = 2\pi r [x]_0^h = 2\pi rh$$

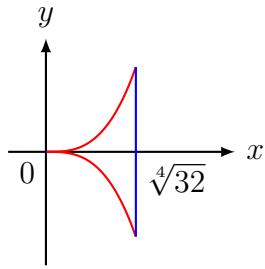
Aufgabe 5.12



$$f(x) = -\frac{r}{h}x + r \quad \Rightarrow \quad f'(x) = -\frac{r}{h}$$

$$\begin{aligned} M &= 2\pi \int_0^h \left(r - \frac{r}{h}x \right) \sqrt{1 + \left(-\frac{r}{h} \right)^2} dx \\ &= 2\pi \sqrt{\frac{h^2 + r^2}{h^2}} \int_0^h \left(r - \frac{r}{h}x \right) dx \\ &= 2\pi \sqrt{\frac{m^2}{h^2}} \left[rx - \frac{r}{2h}x^2 \right]_0^h \\ &= 2\pi \frac{m}{h} \left(rh - \frac{r}{2h}h^2 \right) \\ &= 2\pi \frac{m}{h} \cdot \frac{rh}{2} = \pi rm \end{aligned}$$

Aufgabe 5.13



$$f(x) = \frac{1}{6}x^3 \Rightarrow f'(x) = \frac{1}{2}x^2$$

$$M = 2\pi \int_0^{\sqrt[4]{32}} \frac{1}{6}x^3 \sqrt{1 + \frac{1}{4}x^4} dx = \dots$$

$$\text{Substitution: } u = 1 + \frac{1}{4}x^4$$

$$\frac{du}{dx} = x^3 \Rightarrow dx = \frac{1}{x^3} du$$

$$\text{Grenzen: } u(0) = 1 + 0 = 1$$

$$u(\sqrt[4]{32}) = 1 + 8 = 9$$

$$\begin{aligned} \dots &= 2\pi \int_1^9 \frac{1}{6}x^3 \sqrt{u} \frac{1}{x^3} du \\ &= \frac{1}{3}\pi \int_1^9 \sqrt{u} du \\ &= \frac{2}{9}\pi [u^{3/2}]_1^9 = \frac{2}{9}\pi(27 - 1) = \frac{52}{9}\pi \end{aligned}$$

Aufgabe 4.1

$$(a) \lim_{a \rightarrow \infty} \int_1^a \frac{1}{x^4} dx = \lim_{a \rightarrow \infty} \left[-\frac{1}{3x^3} \right]_1^a$$

$$= \lim_{a \rightarrow \infty} \left(-\frac{1}{3a^3} + \frac{1}{3} \right) = \frac{1}{3}$$

$$(b) \lim_{a \rightarrow \infty} \int_1^a \frac{1}{x^{1.1}} dx = \lim_{a \rightarrow \infty} \int_1^a x^{-1.1} dx = \lim_{a \rightarrow \infty} [-10x^{-0.1}]_1^a$$

$$= \lim_{a \rightarrow \infty} \left(-\frac{10}{a^{0.1}} + \frac{10}{1^{0.1}} \right) = \lim_{a \rightarrow \infty} \left(10 - \frac{10}{a^{0.1}} \right) = 10$$

$$(c) \lim_{a \rightarrow \infty} \int_3^a \frac{4+t}{t^3} dt = \lim_{a \rightarrow \infty} \int_3^a \left(\frac{4}{t^3} + \frac{1}{t^2} \right) dt$$

$$= \lim_{a \rightarrow \infty} \left[\frac{-2}{t^2} - \frac{1}{t} \right]_3^a = \lim_{a \rightarrow \infty} \left[\left(-\frac{2}{a^2} - \frac{1}{a} \right) - \left(-\frac{2}{9} - \frac{1}{3} \right) \right]$$

$$= \lim_{a \rightarrow \infty} \left[\frac{5}{9} - \frac{2}{a^2} - \frac{1}{a} \right] = \frac{5}{9}$$

Aufgabe 4.2

$$(a) \lim_{a \rightarrow 0} \int_a^2 \frac{2}{x^2} dx = \lim_{a \rightarrow 0} \left[\frac{-2}{x} \right]_a^2 = \lim_{a \rightarrow 0} \left(-1 + \frac{2}{a} \right)$$

existiert nicht

$$(b) \lim_{a \rightarrow 0} \int_a^4 \frac{2}{\sqrt{t}} dt = \lim_{a \rightarrow 0} [4\sqrt{t}]_a^4 = \lim_{a \rightarrow 0} (8 - 4\sqrt{a}) = 8$$

$$(c) \lim_{a \rightarrow 0} \int_a^4 u^{-\frac{3}{2}} du = \left[-2u^{-\frac{1}{2}} \right]_a^4 = \lim_{a \rightarrow 0} \left(-1 + \frac{2}{\sqrt{a}} \right)$$

existiert nicht

$$(d) \lim_{a \rightarrow 0} \int_a^8 u^{-\frac{2}{3}} du = \lim_{a \rightarrow 0} \left[3u^{\frac{1}{3}} \right]_a^8 = \lim_{a \rightarrow 0} \left[6 - 3a^{\frac{1}{3}} \right] = 6$$

Aufgabe 4.3

$$(a) \lim_{a \rightarrow \infty} \int_0^a e^{-x} dx = \lim_{a \rightarrow \infty} [-e^{-x}]_0^a = \lim_{a \rightarrow \infty} (-e^{-a} + e^0)$$

$$= \lim_{a \rightarrow \infty} (1 - e^{-a}) = \lim_{a \rightarrow \infty} (1 - e^{-a}) = 1$$

$$(b) \lim_{a \rightarrow -\infty} \int_a^0 e^{-t} dt = \lim_{a \rightarrow -\infty} [-e^{-t}]_a^0 = \lim_{a \rightarrow -\infty} (-1 + e^{-a})$$

existiert nicht

$$(c) \lim_{a \rightarrow \infty} \int_0^a z e^{-z} dz = \lim_{a \rightarrow \infty} [e^{-z}(-z - 1)]_0^a$$

$$\lim_{a \rightarrow \infty} [e^{-a}(-a - 1) - e^0(0 - 1)] = 1$$

$$(d) \lim_{a \rightarrow \infty} \int_0^a y^2 e^{-y} dy = \lim_{a \rightarrow \infty} [e^{-y}(-y^2 - 2y - 1)]_0^a$$

$$= \lim_{a \rightarrow \infty} [e^{-a}(-a^2 - 2a - 1) - (-1)] = 2$$

Aufgabe 4.4

$$(a) \lim_{a \rightarrow -\infty} \int_a^{-2} \frac{1}{(y+1)^3} dy = \lim_{a \rightarrow -\infty} \left[\frac{-1}{2} \cdot \frac{1}{y+1} \right]_a^{-2}$$

$$= \lim_{a \rightarrow -\infty} \left[-\frac{1}{2} + \frac{1}{2} \frac{1}{(a+1)^2} \right] = -\frac{1}{2}$$

$$(b) \lim_{a \rightarrow -1} \int_a^3 \frac{1}{z+1} dz = \lim_{a \rightarrow -1} [\ln(z+1)]_a^3$$

$$= \lim_{a \rightarrow -1} [\ln 4 - \ln(a+1)]$$

existiert nicht

$$(c) \lim_{a \rightarrow \infty} 2 \int_{-a}^a \frac{1}{z^2 + 1} dz = 2 \lim_{a \rightarrow \infty} [\arctan(z)]_{-a}^a$$

$$= 2 \lim_{a \rightarrow \infty} [\arctan(a) - \arctan(-a)] = 2 \left(\frac{\pi}{2} + \frac{\pi}{2} \right) = 2\pi$$