Aufgabe 4.4

(a) $\displaystyle\int_{-3}^{-2}\big(x^3+6x\big)\,\mathrm{d}x=\big[\tfrac{1}{4}x^4+3x^2\big]_{-3}^{-2}=16-\tfrac{189}{4}=-\tfrac{125}{4}=-31.25$

(b) $\displaystyle\int_{-2}^{1}\big(x^4-5\big)\,\mathrm{d}x=\big[\tfrac{1}{5}x^5-5x\big]_{-2}^{1}=-\tfrac{24}{5}-\tfrac{18}{5}=-\tfrac{42}{5}=-8.4$

(c) $\displaystyle\int_{1}^{4}x^5(2x+5)\,\mathrm{d}x=\int_{1}^{4}\big(2x^6+5x^5\big)\,\mathrm{d}x=\big[\tfrac{2}{7}x^7+\tfrac{5}{6}x^6\big]_{1}^{4}\approx 8093.56$

(d) $\displaystyle\int_{-2}^{2}\big(2x-3\big)^2\,\mathrm{d}x=\int_{-2}^{2}\big(4x^2-12x+9\big)\,\mathrm{d}x=\big[\tfrac{4}{3}x^3-6x^2+9x\big]_{-2}^{2}$

$$=\tfrac{14}{3}-\big(-\tfrac{158}{3}\big)=\tfrac{172}{3}=57.\overline{3}$$

Aufgabe 4.5

(a)
$$\begin{aligned}\int_{1}^{3}\big(1.5x^2+3x+k\big)\,\mathrm{d}x&=17\\ \big[\tfrac{1}{2}x^3+\tfrac{3}{2}x^2+kx\big]_{1}^{3}&=17\\ \big(\tfrac{27}{2}+\tfrac{27}{2}+3k\big)-\big(\tfrac{1}{2}+\tfrac{3}{2}+k\big)&=17\\ 25+2k&=17\\ k&=-4\end{aligned}$$

(b)
$$\begin{aligned}\int_{-1}^{0}\big(3x^2-kx+k\big)\,\mathrm{d}x&=-2\\ \big[x^3-\tfrac{k}{2}x^2+kx\big]_{-1}^{0}&=-2\\ 0-\big(-1-\tfrac{k}{2}-k\big)&=-2\\ 1+\tfrac{3k}{2}&=-2\\ \tfrac{3k}{2}&=-3\\ 3k&=-6\\ k&=-2\end{aligned}$$

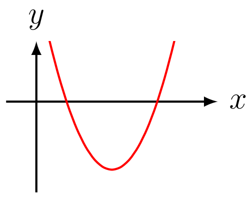
Aufgabe 4.6

(a) $f(x) = x^2 - 5x + 4 = (x-1)(x-4)$

Nullstellen: $x_1 = 1,\ x_2 = 4$

asymptotisches Verhalten: $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} x^2 = +\infty$

$\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} x^2 = +\infty$



$$I = \int_1^4 (x^2 - 5x + 4)\,\mathrm{d}x = \left[\tfrac{1}{3}x^3 - \tfrac{5}{2}x^2 + 4x\right]_1^4 = -4.5$$

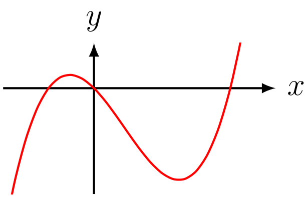
$A = |I| = 4.5$

(b) $f(x) = x^3 - 2x^2 - 3x = x(x^2 - 2x - 3) = x(x-3)(x+1)$

Nullstellen: $x_1 = -1,\ x_2 = 0,\ x_3 = 3$

asymptotisches Verhalten: $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} x^3 = +\infty$

$\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} x^3 = -\infty$



$$I_1 = \int_{-1}^0 f(x)\,\mathrm{d}x = \left[\tfrac{1}{4}x^4 - \tfrac{2}{3}x^3 - \tfrac{3}{2}x^2\right]_{-1}^0 = \tfrac{7}{12}$$

$$I_2 = \int_0^3 f(x)\,\mathrm{d}x = \left[\,\ldots\,\right]_0^3 = -\tfrac{45}{4}$$

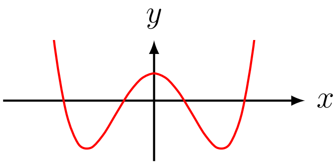
$A = |I_1| + |I_2| = \tfrac{71}{6} = 11.8\overline{3}$

(c) $f(x) = x^4 - 10x^2 + 9 = (x^2-1)(x^2-9) = (x-1)(x+1)(x-9)(x+9)$

Nullstellen: $x_1 = -3,\ x_2 = -1,\ x_3 = 1,\ x_4 = 3$

asymptotisches Verhalten: $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} x^3 = +\infty$

$\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} x^3 = +\infty$



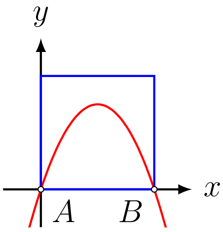
$$I_1 = \int_0^1 f(x)\,\mathrm{d}x = \left[\tfrac{1}{5}x^5 - \tfrac{10}{3}x^3 + \tfrac{9}{2}x\right]_0^1 = \tfrac{88}{15}$$

$$I_2 = \int_1^3 f(x)\,\mathrm{d}x = \left[\ldots\right]_1^3 = -\tfrac{304}{15}$$

$$A = 2(|I_1| + |I_2|) = \tfrac{784}{15} = 52.2\overline{6} \qquad \text{(wegen Symmetrie)}$$

Aufgabe 4.7

$p(x) = 3x - x^2 = x(3 - x) \quad \Rightarrow \quad x_1 = 0,\ x_2 = 3$



Test: $f(1.5) = 1.5(3 - 1.5) = 2.25 \leq 3$ (ok)

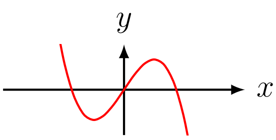
$$A_p = \int_0^3 f(x)\,\mathrm{d}x = \left[\tfrac{3}{2}x^2 - \tfrac{1}{3}x^3\right]_0^3 = 13.5 - 9 - (0 - 0) = 4.5$$

$\frac{1}{2}\overline{AB}^2 = \frac{1}{2} \cdot 9 = 4.5$ (ok)

Aufgabe 4.8

$p(x) = ax - x^3 = x(a - x^2) = x\big(\sqrt{a} - x\big)\big(\sqrt{a} + x\big)$

Nullstellen: $x_1 = -\sqrt{a},\ x_2 = 0,\ x_3 = \sqrt{a}$

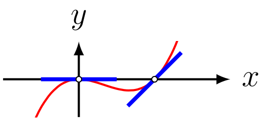


$$\begin{aligned} \int_0^{\sqrt{a}} \big(ax - x^3\big)\,\mathrm{d}x &= 9 \\ \left[\tfrac{1}{2}ax^2 - \tfrac{1}{4}x^4\right]_0^{\sqrt{a}} &= 9 \\ \tfrac{1}{2}a^2 - \tfrac{1}{4}a^2 &= 9 \\ \tfrac{1}{4}a^2 &= 9 \\ a &= 6 \end{aligned}$$

Aufgabe 4.9

Ansatz: $f(x) = ax^3 + bx^2 + cx + d$

$\qquad f'(x) = 3ax^2 + 2bx + c$



$f(1) = 0: \quad a + b + c + d = 0 \quad (1)$

$f'(1) = 1: \quad 3a + 2b + c = 1 \quad (2)$

$f(0) = 0: \quad d = 0 \quad (3)$

$f'(0) = 0: \quad c = 0 \quad (4)$

$$\begin{aligned} a + b &= 0 \\ 3a + 2b &= 1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} a &= 1 \\ b &= -1 \end{aligned} \quad \Rightarrow \quad f(x) = x^3 - x^2$$

$$\int_0^1 f(x)\,\mathrm{d}x = \left[\tfrac{1}{4}x^4 - \tfrac{1}{3}x^3\right]_0^1 = -\tfrac{1}{12} \quad \Rightarrow \quad A = \tfrac{1}{12}$$

Aufgabe 4.13

$$\begin{aligned} p(x) &= 2x^3 - 6x + a \\ p'(x) &= 6x^2 - 6 \\ p''(x) &= 12x \end{aligned}$$

(a) Tiefstellen:

$$\begin{aligned} p'(x) &= 0 \\ 6x^2 - 6 &= 0 \\ 6(x^2 - 1) &= 0 \end{aligned}$$

$x_1 = -1 \;\Rightarrow\; p''(-1) = -12 < 0 \;\Rightarrow\; \mathrm{HoP}(-1, 4 + a)$

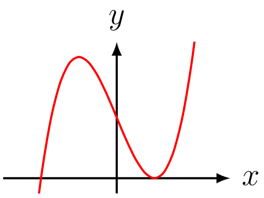
$x_2 = 1 \;\Rightarrow\; p''(1) = 12 > 0 \;\Rightarrow\; \mathrm{TiP}(1, -4 + a)$

Die Parabel p soll die x-Achse in $\mathrm{TiP}(1, -4 + a)$ berühren.

$\Rightarrow a = 4 \quad \Rightarrow \quad p(x) = 2x^3 - 6x + 4$

(b) $p(x) = 2x^3 - 6x + 4$

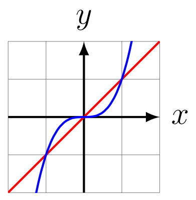
Nullstellen: $x_1 = x_2 = 1$, $x_3 = -2$; Ordinatenabschnitt: $y_0 = 4$



$$A = \int_{-2}^{1} \left(2x^3 - 6x + 4\right) \mathrm{d}x = \left[\tfrac{1}{2}x^4 - 3x^2 + 4x\right]_{-2}^{1} = 13.5$$

Aufgabe 4.14

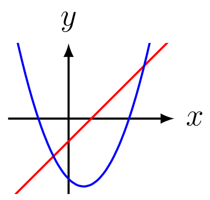
(b) $f_1\colon y = x \qquad f_2\colon y = x^3$



$$A = 2\int_0^1 \left(x - x^3\right) = 2\left[\tfrac{1}{2}x^2 - \tfrac{1}{4}x^4\right]_0^1 = 2\cdot\left(\tfrac{1}{2} - \tfrac{1}{4}\right) = \tfrac{1}{2}$$

(c) $f_1\colon y = 2x - 3 \qquad f_2\colon y = x^2 - 2x - 8$

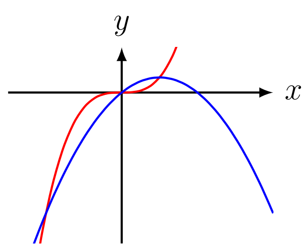
$$2x - 3 = x^2 - 2x - 8$$
$$0 = x^2 - 4x - 5 = (x-5)(x+1)$$
$$x_1 = -1 \quad\Rightarrow\quad y_1 = -5$$
$$x_2 = 5 \quad\Rightarrow\quad y_2 = 7$$



$$A = \int_{-1}^{5} \left(2x - 3 - \left(x^2 - 2x - 8\right)\right)\,\mathrm{d}x$$
$$= \int_{-1}^{5} \left(-x^2 + 4x + 5\right)\,\mathrm{d}x = \left[-\tfrac{1}{3}x^3 + x^2 + 5x\right]_{-1}^{5} = 36$$

(d) $f_1\colon y = x^3 \qquad f_2\colon y = 2x - x^2$

$$x^3 = 2x - x^2$$
$$0 = x^3 + x^2 - 2x = x(x^2 + x - 2) = x(x-1)(x+2)$$
$$x_1 = 0 \quad\Rightarrow\quad y_1 = 0$$
$$x_2 = 1 \quad\Rightarrow\quad y_2 = 1$$
$$x_3 = -2 \quad\Rightarrow\quad y_3 = -8$$



$$A_1 = \int_{-2}^{0} \left(x^3 + x^2 - 2x\right) \mathrm{d}x \quad = \left[\tfrac{1}{4}x^4 + \tfrac{1}{3}x^3 - x^2\right]_{-2}^{0} = \tfrac{8}{3}$$

$$A_2 = \int_{1}^{0} \left(x^3 + x^2 - 2x\right) \mathrm{d}x \quad = \left[\tfrac{1}{4}x^4 + \tfrac{1}{3}x^3 - x^2\right]_{1}^{0} = \tfrac{5}{12}$$

$$A = A_1 + A_2 = \tfrac{37}{12}$$

Aufgabe 4.15

$$p(x) = x^3 - 3x^2 + 6$$

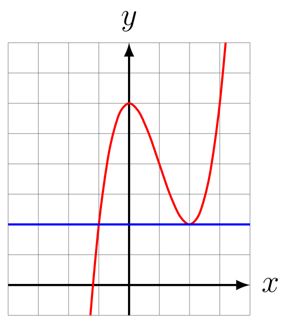
$$p'(x) = 3x^2 - 6x = 3x(x-2)$$

$$p''(x) = 6x - 6 = 6(x-1)$$

(a) $p'(x) = 0$

$$x_1 = 0 \quad \Rightarrow \quad p''(0) = -6 < 0 \quad \Rightarrow \quad \mathrm{HoP}(0,6)$$

$$x_2 = 2 \quad \Rightarrow \quad p''(2) = 6 > 0 \quad \Rightarrow \quad \mathrm{TiP}(2,2)$$



(b) $t\colon y = 2$ (parallel zur x-Achse durch TiP)

$$p(x) = t(x)$$

$$x^3 - 3x^2 + 6 = 2$$

$$x^3 - 3x^2 + 4 = 0$$

$$x_1 = x_2 = 2$$

$$x_3 = -1$$

$$A = \int_{-1}^{2} \big(f(x) - t(x)\big) \,\mathrm{d}x = \int_{-1}^{2} \left(x^3 - 3x^2 + 4\right) \mathrm{d}x$$

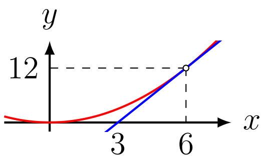
$$= \left[\tfrac{1}{4}x^4 - x^3 + 4x\right]_{-1}^{2} = \tfrac{27}{4} = 6.75$$

Aufgabe 4.19

$p(x) = \frac{1}{3}x^2$; $p'(x) = \frac{2}{3}x$

Kurventangente in $P(6, y)$ (z. B. via Taylorpolynom)

$t(x) = f(6) + f'(6)(x - 6) = 12 + 4(x - 6) = 4x - 12$

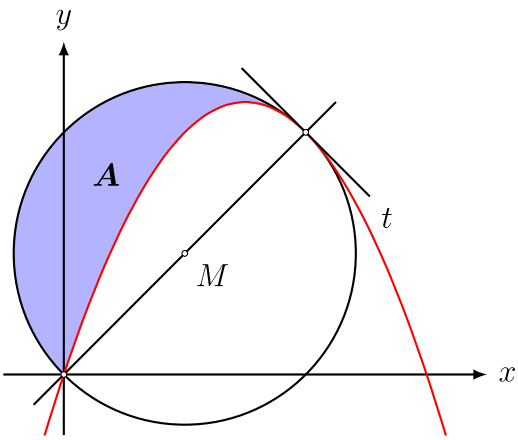


Nullstelle von t: $x = 3$

$$A = A_{\text{Parabel}} - A_{\text{Dreieck}} = \int_0^6 \tfrac{1}{3}x^2\,\mathrm{d}x - \tfrac{1}{2} \cdot 3 \cdot 12 -$$

$$= \left[\tfrac{1}{9}x^3\right]_0^6 - 18 = 24 - 18 = 6$$

Aufgabe 4.38



Kreis mit $M(2, 2)$ und $r = \sqrt{2^2 + 2^2} = 2\sqrt{2}$

Ansatz: $f(x) = ax^2 + bx \quad (c = 0 \text{ da } (0, 0) \in G_f)$

$f'(x) = 2ax + b$

$f(4) = 4$: $16a + 4b = 4 \quad \Rightarrow \quad 4a + b = 1 \quad (1)$

$f'(4) = -1$: $8a + b = -1 \quad (2)$

(1) (2) $\Rightarrow \quad f(x) = -\frac{1}{2}x^2 + 3x$

Gleichung des Durchmessers: $g(x) = x$

Fläche zwischen Diagonale und Parabel:

$$I = \int_0^4 \big(f(x) - g(x)\big)\,\mathrm{d}x = \int_0^4 \left(-\tfrac{1}{2}x^2 + 3x - x\right)\mathrm{d}x$$

$$= \int_0^4 \left(-\tfrac{1}{2}x^2 + 2x\right)\mathrm{d}x = \left[-\tfrac{1}{6}x^3 - x^2\right]_0^4 = \tfrac{16}{3}$$

$A = \frac{1}{2}A_K - I = 4\pi - \frac{16}{3} \approx 7.23$