Integralrechnung (Partialbruchzerlegung) Lösungen+ Übungen

Aufgabe 1

$$\int_{-1}^{0} \frac{u+1}{u-1}\,\mathrm{d}u = \dots$$

Polynomdivision: $(u+1):(u-1) = 1 + \dfrac{2}{u-1}$

$$\begin{aligned} \dots &= \int_{-1}^{0} \left(1 + \frac{2}{u-1}\right)\mathrm{d}u = \big[u + 2\ln|u-1|\big]_{-1}^{0} \\ &= \big(0 + 2\ln(1)\big) - \big(-1 + 2\ln(2)\big) = 1 - 2\ln(2) \end{aligned}$$

Aufgabe 2

Partialbruchzerlegung: $\dfrac{1}{t^2-1} = \dfrac{A}{t-1} + \dfrac{B}{t-1}$

$$1 = A(t+1) + B(t-1)$$

$$1 = \underbrace{(A+B)}_{0}\,t + \underbrace{(A-B)}_{1}$$

$\Rightarrow A = \frac{1}{2},\ B = -\frac{1}{2}$

$$\int_{2}^{3} \frac{1}{t^2-1}\,\mathrm{d}t = \frac{1}{2}\int_{2}^{3}\left(\frac{1}{t-1} - \frac{1}{t+1}\right)\mathrm{d}t$$

$$= \frac{1}{2}\big[\ln|t-1| - \ln|t+1|\big]_{2}^{3} = \frac{1}{2}(\ln 2 - \ln 4 - \ln 1 + \ln 3)$$

$$= \frac{1}{2}\ln\frac{2\cdot 3}{4} = \frac{1}{2}\ln\frac{3}{2}$$

Beachte: $\log_a(x\cdot y) = \log_a(a) + \log_a(b)$
$\log_a(x:y) = \log_a(a) - \log_a(b)$

Aufgabe 3

$$\int_{-1}^{1} \frac{13-x}{x^2-x-6}\,\mathrm{d}x = \int_{-1}^{1} \frac{13-x}{(x+2)(x-3)}\,\mathrm{d}x = \dots$$

Partialbruchzerlegung: $\dfrac{13-x}{(x+2)(x-3)} = \dfrac{A}{x+2} + \dfrac{B}{x-3}$

$$13 - x = A(x-3) + B(x+2)$$

$$-x + 13 = \underbrace{(A+B)}_{-1}\,x + \underbrace{(-3A+2B)}_{13}$$

$\Rightarrow A = -3,\ B = 2$

$$\ldots = \int_{-1}^{1} \left(\frac{2}{x-3} - \frac{3}{x+2} \right) \mathrm{d}x = \big[2\ln|x-3| - 3\ln|x+2| \big]_{-1}^{1}$$

$$= 2\ln(2) - 3\ln(3) - 2\ln(4) + 3\ln(1) = 2\ln(\tfrac{1}{2}) - 3\ln(3)$$

$$= \ln(\tfrac{1}{4}) - \ln(27) = \ln(\tfrac{1}{108}) = -\ln(108)$$

Aufgabe 4

$$\int_{2}^{4} \frac{z^2+1}{z^2-1} \,\mathrm{d}z = \int_{2}^{4} \frac{z^2+1}{(z-1)(z+1)} \,\mathrm{d}z = \ldots$$

Polynomdivision: $(z^2+1) : (z^2-1) = 1 + \dfrac{2}{z^2-1}$

Partialbruchzerlegung: $\dfrac{2}{(z-1)(z+1)} = \dfrac{A}{z-1} + \dfrac{B}{z+1}$

$$2 = A(z+1) + B(z-1)$$
$$2 = \underbrace{(A+B)}_{0} z + \underbrace{(A-B)}_{2}$$

$\Rightarrow \quad A = 1,\ B = -1$

$$\ldots = \int_{2}^{4} \left(1 + \frac{1}{z-1} - \frac{1}{z+1} \right) \mathrm{d}z = \big[z + \ln|z-1| - \ln|z+1| \big]_{2}^{4}$$

$$= 4 + \ln(3) - \ln(5) - 2 - \ln(1) + \ln(3) = 2 + \ln\left(\tfrac{9}{5}\right)$$

Aufgabe 5

$2 + \ln(18)$

Aufgabe 6

Polynomdivision: $(3u^2 + 3u - 4) : (u + 2u + 1) = 3 - \dfrac{3u+7}{(u+1)^2}$

Partialbruchzerlegung: $\dfrac{3u+7}{(u+1)^2} = \dfrac{A}{u+1} + \dfrac{B}{(u+1)^2} \quad || \cdot \mathrm{HN}$

$$3u + 7 = A(u+1) + B$$
$$3u + 7 = Au + (A+B)$$
$$3 = A$$
$$7 = A + B \quad \Rightarrow \quad B = 4$$

$$\int_{0}^{2} \frac{3u^2+3u-4}{(u+1)^2} \,\mathrm{d}u = \int_{0}^{2} \left(3 - \frac{3}{u+1} - \frac{4}{(u+1)^2} \right) \mathrm{d}u$$

$$= \left[3u - 3\ln|u+1| + \frac{4}{u+1} \right]_{0}^{2} = 6 - 3\ln(3) + \tfrac{4}{3} - 4 = \tfrac{10}{3} - 3\ln(3)$$

Aufgabe 7

$$\int_{-3}^{-1} \frac{y+8}{12y-3y^2}\,\mathrm{d}y = \frac{1}{3}\int_{-3}^{-1} \frac{y+8}{y(4-y)}\,\mathrm{d}y = \ldots$$

Partialbruchzerlegung: $\dfrac{y+8}{y(4-y)} = \dfrac{A}{y} + \dfrac{B}{4-y}$

$$y+8 = A(4-y) + By$$

$$y+8 = \underbrace{(-A+B)}_{1}y + \underbrace{4A}_{8}$$

$\Rightarrow \quad A=2,\ B=3$

$$\ldots \overset{!}{=} \frac{1}{3}\int_{-3}^{-1}\left(\frac{2}{y} - \frac{3}{y-4}\right)\mathrm{d}y = \frac{1}{3}\big[2\ln|y| - 3\ln|y-4|\big]_{-3}^{-1}$$

$$= \frac{1}{3}\big(2\ln(1) - 3\ln(5) - 2\ln(3) + 3\ln(7)\big)$$

$$= \ln(7) - \ln(5) - \frac{2}{3}\ln(3) = \ln\left(\frac{7}{5}\right) - \frac{2}{3}\ln(3)$$

Aufgabe 8

$q(x) = x^8 - x^7 - x^6 + 3x^5 - 7x^4 + 9x^3 - 7x^2 + 5x - 2$

x_0		-1	-1	3	-7	9	-7	5	-2
1	1	0	-1	2	-5	4	-3	2	0
1	1	1	0	2	-3	1	-2	0	
1	1	2	2	4	1	2	0		
-2	1	0	2	0	1	0			

$q(x) = (x-1)^3(x+2)(x^4+2x^2+1) = (x-1)^3(x+2)(x^2+1)^2$

$$\frac{1}{(x+2)(x-1)^3(x^2+1)^2} = \frac{A_{11}}{x+2} + \frac{A_{21}}{x-1} + \frac{A_{22}}{(x-1)^2} + \frac{A_{23}}{(x-1)^3} + \frac{B_{11}x + C_{11}}{x^2+1} + \frac{B_{12}x + C_{12}}{(x^2+1)^2}$$