Aufgabe 1 (5P)

		monoton		
	beschränkt	wachsend	fallend	alternierend
$a_n = 1 + 0.5^n$	☒	☐	☒	☐
$b_n = n^2 - 10n$	☐	☐	☐	☐
$c_n = (-2)^n$	☐	☐	☐	☒
$d_n = \dfrac{n+2}{n+3}$	☒	☒	☐	☐
$e_n = 4 + \dfrac{(-1)^n}{n}$	☒	☐	☐	☐

(a_n): 1.5, 1.25, 1.125, 1.0625, 1.03125, 1.015625, 1.0078125, ...

(b_n): −9, −16, −21, −24, −25, −24, −21, −16, −9, 0, 11, 24, 39, ...

(c_n): −2, 4, −8, 16, −32, 64, −128, 256, ...

(d_n): 0.75, 0.8, $0.8\overline{3}$, $0.8\overline{571428}$, 0.875, $0.\overline{8}$, 0.9, $0.9\overline{09}$, ...

(e_n): 3.0, 4.5, $3.\overline{6}$, 4.25, 3.8, $4.1\overline{6}$, $3.8\overline{571428}$, 4.125, $3.\overline{8}$, 4.1, $3.9\overline{09}$, ...

Aufgabe 2 (4P)

(a) $a_n = \dfrac{(10n - 4n^2 + 5)/n^2}{(7 + 2n - 3n^2)/n^2} = \dfrac{10/n - 4 + 5/n^2}{7/n^2 + 2/n - 3} \xrightarrow{n\to\infty} \dfrac{-4}{-3} = \dfrac{4}{3}$

(b) $a_n = \sqrt{n+3} - \sqrt{n} = \dfrac{\left(\sqrt{n+3} - \sqrt{n}\right)\left(\sqrt{n+3} + \sqrt{n}\right)}{\sqrt{n+3} + \sqrt{n}}$

$$= \frac{n + 3 - n}{\sqrt{n+3} + \sqrt{n}} = \frac{3}{\sqrt{n+3} + \sqrt{n}} \xrightarrow{n\to\infty} 0$$

Aufgabe 3 (4P)

(a) $\lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{9n+1}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{9n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{3\sqrt{n}} = \frac{1}{3}$

(b) $\lim_{n\to\infty} \sqrt[n]{n} = 1$

(c) $\lim_{n\to\infty} \left(-\frac{4}{3}\right)^n$ existiert nicht

(d) $\lim_{n\to\infty} \frac{2^n}{n^2} = \infty$

(e) $\lim_{n\to\infty} \frac{\ln(n)}{2n} = \frac{1}{2} \lim_{n\to\infty} \frac{\ln(n)}{n} = 0$

(f) $\lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = \cos(0) = 1$

(g) $\lim_{n\to\infty} \left(1 - \frac{4}{n}\right)^n = \mathrm{e}^{-4}$

(h) $\lim_{n\to\infty} \frac{2^{n+1}+1}{2^n} = \lim_{n\to\infty} \frac{2 \cdot 2^n}{2^n} = 2$

Aufgabe 4 (6P)

(a) $\lim_{x\to 2} \frac{x^2-2x}{x-2} = \lim_{x\to 2} \frac{x(x-2)}{x-2} = \lim_{x\to 2} x = 2$

(b) $\lim_{x\to -3} \frac{x^2+8x+15}{x+3} = \lim_{x\to -3} \frac{(x+3)(x+5)}{x+3} = \lim_{x\to -3} (x+5) = 2$

(c) $\lim_{x\to 1^+} \frac{1}{1-x} = -\infty$

(d) $\lim_{x\to -\infty} 2^x = 0$

(e) $$\lim_{x\to 0} \left(\frac{1}{2x+x^2} - \frac{1}{2x}\right) = \lim_{x\to 0} \frac{2-(2+x)}{2x(2+x)}$$
$$= \lim_{x\to 0} \frac{-x}{2x(2+x)} = \lim_{x\to 0} \frac{-1}{2(2+x)} = -\frac{1}{4}$$

Aufgabe 5 (4P)

Vermutung: $a = \lim\limits_{n\to\infty} \frac{2n}{n+1} = 2$;

Sei $\varepsilon > 0$ vorgegeben.

$$
\begin{aligned}
|a_n - a| &< \varepsilon \\
\left|\frac{2n}{n+1} - 2\right| &< \varepsilon \\
\left|\frac{2n}{n+1} - \frac{2(n+1)}{n+1}\right| &< \varepsilon \\
\left|\frac{2n - (2n+2)}{n+1}\right| &< \varepsilon \\
\left|\frac{-2}{n+1}\right| &< \varepsilon \quad \text{(Betrag anwenden)}
\end{aligned}
$$

$$\frac{2}{n+1} < \varepsilon$$

$$\frac{2}{\varepsilon} < n+1$$

$$\frac{2}{\varepsilon} - 1 < n \quad \text{oder:} \quad \frac{2-\varepsilon}{\varepsilon} < n$$

Setzen wir $n_\varepsilon = \frac{2}{\varepsilon} - 1$, so gilt $\left|\frac{2n}{n+1} - 2\right| < \varepsilon$ für alle $n > n_\varepsilon$.

Aufgabe 6 (4P)

$$\frac{a_1}{1-q} \overset{(1)}{=} 15 \text{ und } \frac{a_1^2}{1-q^2} \overset{(2)}{=} 25$$

(1) $\Rightarrow$ $a_1 = 15(1-q)$ und in (2) einsetzen:

$$
\begin{aligned}
\frac{(15(1-q))^2}{1-q^2} &= 25 \\
225(1-q)^2 &= 25(1-q^2) \\
9(1-q)^2 &= (1-q)(1+q) \\
9(1-q) &= 1+q \\
9 - 9q &= 1+q \\
8 &= 10q \\
q &= 0.8 \\
a_1 &= 15(1-0.8) = 3
\end{aligned}
$$